1. Which of the following is an inner product? Explain.

   1.1. (5 points) For any $(v^1, v^2), (w^1, w^2) \in \mathbb{R}^2$,

   $$(v, w) = v^1 w^2 + v^2 w^1.$$

   1.2. (5 points) For any $(v^1, v^2), (w^1, w^2) \in \mathbb{R}^2$,

   $$(v, w) = 2v^1 w^1 - v^1 w^2 - v^2 w^1 + 2v^2 w^2.$$

2. Let V be a 2-dimensional real vector space and (e_1, e_2) be a basis. Which of the following 2-tensors is an inner product? Explain.

   2.1. (5 points) Let $T : V \times V \to \mathbb{R}$ be the 2-tensor given by

   $$\begin{aligned} T(e_1, e_1) &= 1 \\ T(e_1, e_2) = T(e_2, e_1) &= 1 \\ T(e_2, e_2) &= 1 \end{aligned}$$

   2.2. (5 points) Let $T : V \times V \to \mathbb{R}$ be given by

   $$\begin{aligned} T(e_1, e_1) &= 1 \\ T(e_1, e_2) &= 1 \\ T(e_2, e_1) &= -1 \\ T(e_2, e_2) &= 1 \end{aligned}$$

3. (10 points) Without relying on the lectures, prove that if V_1, V_2 are subspaces of V and

   $$V_1 \cap V_2 = \{0\},$$

   then if $v_1 \in V_1$ and $v_2 \in V_2$ are nonzero vectors, then $\{v_1, v_2\}$ is a linearly independent set.

4. (30 points) Let V be a real inner product space and let $v, w \in V$ be nonzero vectors. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

   $$f(t) = |v - tw|^2.$$

   4.1. (10 points) Prove that f has a unique mininum.

4.2. (10 points) Find the minimum value of f and the value of t for which $f(t)$ equals the mininum value.

4.3. (10 points) Use your solution to the previous parts of this problem to prove that for any $v, w \in V$,

$$(v, w)^2 \leq (v, v)(w, w)$$

and equality holds if and only if there exists a scalar $s \in \mathbb{R}$ such that

$$v = sw \text{ or } w = sv.$$