MA-GY 7043: Linear Algebra II

Course Requirements
Notation
Abstract Vector Spaces
Abstract Matrices
Change of Basis
Linear Functions and Maps

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Outline I

Assignments

- All homework assignments and exams will be handled using Gradescope
- Homework
  - Every one or two weeks
  - Provided as Overleaf project and Gradescope assignment
  - Solutions must be typed up using LaTeX
  - Submissions uploaded as PDF to Gradescope
- Midterm and Final
  - In person
  - Format to be determined
    - 150 minute written exam
    - 30 minute oral exam

Grading Policy

- Course grade
  - Homework: 20%
  - Midterm: 30%
  - Final: 50%
  - Tweaks
- Homework and Exams
  - Partial credit for correct and relevant logical reasoning
  - Full credit for correct and relevant logical reasoning and correct answer
  - No credit for correct answer but incorrect logical reasoning
  - Incorrect logic and calculations wil be severely penalized

Course Information

- Web Pages
  - My homepage
  - Course Homepage
  - Course Calendar
- Textbook
  - Yisong Yang, **A Concise Text on Advanced Linear Algebra**, Cambridge University Press
  - PDF available in Ed Discussion Resources

Functions and Maps

- We will use the following notation when defining a function or map:

$$\begin{aligned} \textit{function} : \textit{domain} &\to \textit{codomain} \\ \textit{input} &\mapsto \textit{output} \end{aligned}$$

- When doing calculations and proofs, It is important to keep track of the domain and codomain of a function
- Given maps $F : X \to Y$ and $G : W \to Z$, then F can be composed with G,

$$G \circ F : X \to Z$$

  if and only if $Y \subset W$,
- If you make sure that each input to a function really is an element of the domain and each output really is treated as an element of the codomain, you will catch 90% of your errors

Logical Symbols

- $\forall$ means *for each* or *for any* or *for all*
- $\exists$ means *there is at least one* or *there exists at least one*
- $\exists!$ means *there is exactly one* or *there exists exactly one*
- *(assumption)* $\implies$ *(conclusion)* means
  - *if (assumption), then (conclusion)*
  - *(assumption) only if (conclusion)*
  - *(conclusion) if (assumption)*
- $\iff$ means *if and only if*

Abstract Vector Space

- Let $\mathbb{F}$ be either the reals (denoted $\mathbb{R}$) or the complex numbers (denoted $\mathbb{C}$)
- A vector space over $\mathbb{F}$ is a set V with the following:
  - An element called the **zero vector**, denoted $\vec{0}$, 0_V, or simply 0
  - An operation called **vector addition**:

$$V \times V \to V$$
$$(v_1, v_2) \mapsto v_1 + v_2$$

  - An operation called **scalar multiplication**:

$$V \times \mathbb{F} \to V$$
$$(v, r) \mapsto rv = vr$$

such that the following properties hold

Properties of Vector Addition

- Associativity

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_2)$$

- Commutativity

$$v_1 + v_2 = v_2 + v_1$$

- Identity element:

$$v + \vec{0} = v$$

- Inverse element: For each $v \in V$, there exists an element, denoted $-v$, such that

$$v + (-v) = \vec{0}$$

Scalar Multiplication

- Properties
  - Associativity

$$(f_1 f_2)v = f_1(f_2 v)$$

  - Distributivity

$$(f_1 + f_2)v = f_1 v + f_2 v$$
$$f(v_1 + v_2) = fv_1 + fv_2$$

  - Identity element

$$1v = v$$

Consequences

- $$\begin{aligned} 0v &= 0v + v - v \\ &= 0v + 1v - v \\ &= (0+1)v - v \\ &= v - v \\ &= \vec{0} \end{aligned}$$
- $$\begin{aligned} (-1)v &= (-1)v + v - v \\ &= (-1)v + 1v - v \\ &= (-1+1)v - v \\ &= 0v - v \\ &= \vec{0} - v \\ &= -v \end{aligned}$$

Valid and Invalid Expressions

- Valid expressions

$$\begin{aligned}(\text{vector}) + (\text{vector})\\(\text{scalar}) + (\text{scalar})\\(\text{scalar})(\text{vector})\\(\text{vector})(\text{scalar})\\(\text{scalar})(\text{scalar})\end{aligned}$$

- Invalid expressions

$$\begin{aligned}(\text{vector}) + (\text{scalar})\\(\text{scalar}) + (\text{vector})\\(\text{vector})(\text{vector})\end{aligned}$$

Linear Combination of Vectors

- Given a finite set of vectors $v_1, \ldots, v_m \in V$ and scalars $f^1, \ldots, f^m$, the vector

$$f^1 v_1 + \cdots + f^m v_m$$

is called a **linear combination** of $v_1, \ldots, v_m$

- Given a subset $S \subset V$, not necessarily finite, the **span** of S is the set of all possible linear combinations of vectors in S

$$[S] = \{f^1 v_1 + \cdots + f^m v_m \ : \forall\ f^1, \ldots, f^m \in \mathbb{F} \text{ and } v_1, \ldots, v_m \in S\}$$

- A vector space V is **finite dimensional** if there is a finite set S of vectors such that

$$[S] = V$$

- *In this course vector spaces are assumed to be finite dimensional*

Basis of a Vector Space

- A set $\{v_1, \ldots, v_k\} \subset V$ is **linearly independent** if
$$f^1 v_1 + \cdots f^m v_m = \vec{0} \implies f^1 = \cdots = f^m = 0,$$
- A finite set $S = (v_1, \ldots, v_m) \subset V$ is called a **basis** of V if it is linearly independent and
$$[S] = V$$
- For such a basis, if $v \in V$, then there exist a unique set of scalar coefficients $(a^1, \ldots, a^m)$ such that
$$v = a^k v_k$$
- In other words, the map
$$\mathbb{F}^m \to V$$
$$\langle f^1, \ldots, f^m \rangle \mapsto f^1 v_1 + \cdots + f^m v_m$$
is bijective

Dimension of a Vector Space

- Every finite dimensional vector space has a basis
- Any two bases have the same number of elements
- The dimension of a vector space is defined to be the number of elements in a basis
- The dimension of V is denoted $\dim V$

Definition of an Abstract Matrix

- An m-by-n **abstract matrix** M is a table of symbols with m rows and n columns
- The element in the j-th row and k-th column is labeled

$$M^j_k$$

- Therefore,

$$M = \begin{bmatrix} M^1_1 & \cdots & M^1_n \\ \vdots & & \vdots \\ M^m_1 & \cdots & M^m_n \end{bmatrix}$$

Row and Column Matrices

- A **row matrix** is a matrix with 1 row,

$$R = \begin{bmatrix} R_1 & \cdots & R_n \end{bmatrix}$$

- A **column matrix** is a matrix with 1 column

$$C = \begin{bmatrix} C^1 \\ \vdots \\ C^m \end{bmatrix}$$

Product of Row and Column Matrices (Part 1)

- Let R be a row matrix with m columns and C be a column matrix with m rows,

$$R = \begin{bmatrix} R_1 & \cdots & R_m \end{bmatrix} \text{ and } C = \begin{bmatrix} C^1 \\ \vdots \\ C^m \end{bmatrix}$$

- Suppose that for each $1 \leq k \leq m$, the product

$$R_j C^j$$

is well defined, e.g.,

$$R_1, \ldots, R_m, C^1, \ldots, C^m \in \mathbb{F} \tag{1}$$

$$R_1, \ldots, R_m \in V \text{ and } C^1, \ldots, C^m \in \mathbb{F} \tag{2}$$

$$R_1, \ldots, R_m \in \mathbb{F} \text{ and } C^1, \ldots, C^m \in V \tag{3}$$

Product of Row and Column Matrices (Part 2)

- The **matrix product** of R and C is defined to be the 1-by-1 matrix

$$RC = \begin{bmatrix} R_1 & \cdots & R_m \end{bmatrix} \begin{bmatrix} C^1 \\ \vdots \\ C^m \end{bmatrix} = R_1 C^1 + \cdots + R_m C^m$$

- If (1) holds, then RC is a scalar-valued 1-by-1 matrix
- If (2) or (3) holds, then RC is a vector-valued 1-by-1 matrix

Product of Two Matrices

- Let $R^1, \ldots, R^m$ denote the rows of an m-by-k matrix

$$M = \begin{bmatrix} M^1_1 & \cdots & M^1_k \\ \vdots & & \vdots \\ M^m_1 & \cdots & M^m_k \end{bmatrix} = \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix}$$

- Let $C_1, \ldots, C_n$ denote the columns of a k-by-n matrix

$$N = \begin{bmatrix} N^1_1 & \cdots & N^1_n \\ \vdots & & \vdots \\ N^k_1 & \cdots & N^n_m \end{bmatrix} = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}$$

- The product of M and N is defined to be the m-by-n matrix, denoted MN, where for each

$$1 \leq j \leq m \text{ and } 1 \leq k \leq n,$$

the element in the j-th row and k-th column is

$$(MN)^j_k = R^j C_k$$

Properties of Abstract Matrix Multiplication

- If A, B are m-by-k matrices and C is a k-by-n matrix, then

$$(A + B)C = AC + BC$$

- If A is an m-by-k matrix and B, C are k-by-n matrices, then

$$A(B + C) = AB + AC$$

- If A is an m-by-j matrix, B is a j-by-k matrix, and C is a k-by-n matrix, then

$$(AB)C = A(BC)$$

Matrix Notation for Vector with Respect to Basis

- Let $(b_1, \ldots, b_m)$ be a basis of a vector space V
- For each $v \in V$, there are unique coefficients $c^1, \ldots, c^m \in \mathbb{F}$ such that

$$\begin{aligned} v &= b_1 c^1 + \cdots + b_m c^m \\ &= \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} \begin{bmatrix} c^1 \\ \vdots \\ c^m \end{bmatrix} \\ &= BC, \end{aligned}$$

where the basis is written as a row matrix of vectors

$$B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}$$

and the coefficients are written as a column matrix of scalars

$$C = \begin{bmatrix} c^1 \\ \vdots \\ c^m \end{bmatrix}$$

Matrices of Matrices

- Let M be an abstract m-by-k matrix

$$M = \begin{bmatrix} M_1^1 & \cdots & M_k^1 \\ \vdots & & \vdots \\ M_1^m & \cdots & M_k^m, \end{bmatrix}$$

where each M_j^i is itself an p-by-p matrix

- Therefore, M is an mp-by-kp matrix, broken up into p-by-p blocks
- Let N be an abstract k-by-n matrix

$$N = \begin{bmatrix} N_1^1 & \cdots & N_n^1 \\ \vdots & & \vdots \\ N_1^k & \cdots & N_n^k, \end{bmatrix}$$

where each N_l^j is itself an p-by-p matrix

- Then the abstract matrix product $A = MN$ is the same as the standard matrix product $A = MN$

Change of Basis of Formula

- Let $E = (e_1, \ldots, e_n)$ be a basis of V and

$$v = a^1 e_1 + \cdots + a^n e_n$$

- If $F = (f_1, \ldots, f_n)$ is another basis, then there is a unique matrix M such that for each $1 \le k \le n$,

$$f_k = M_k^1 e_1 + \cdots + M_k^n e_n$$

- v can also be written with respect to the basis F,

$$v = b^1 f_1 + \cdots + b^n f_n$$

- How are $(a^1, \ldots, a^n)$ and $(b^1, \ldots, b^n)$ related?

Standard Basis of $\mathbb{F}^3$

- Denote the standard basis vectors of $\mathbb{F}^3$ by

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \; e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \; e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

- The basis can be written as a row matrix of column vectors:

$$E = \begin{bmatrix}e_1 & e_2 & e_3\end{bmatrix} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = I$$

- Any vector $v = (v^1, v^2, v^3) \in \mathbb{F}$ can be written as

$$v = \begin{bmatrix}v^1\\v^2\\v^3\end{bmatrix} = e_1 v^1 + e_2 v^2 + e_3 v^3 = \begin{bmatrix}e_1 & e_2 & e_3\end{bmatrix}\begin{bmatrix}v^1\\v^2\\v^3\end{bmatrix} = Ev$$

Change of Basis Example on $\mathbb{F}^3$

- Consider a basis of $\mathbb{F}^3$,

$$F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

- Given a vector $v = (v^1, v^2, v^3)$, there are coefficients b^1, b^2, b^3 such that

$$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = f_1 b^1 + f_2 b^2 + f_3 b^3$$

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} b^1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} b^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} b^3 = Fb$$

- Therefore,

$$b = F^{-1} v$$

Change of Basis Example on $\mathbb{F}^3$

- Consider a basis

$$F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

- Given a vector $v = (1, 2, 3)$, there are coefficients b^1, b^2, b^3 such that

$$\begin{aligned} (1, 2, 3) &= b^1(1, -1, 1) + b^2(0, 1, 1) + b^3(0, 0, 1) \\ &= (b^1, -b^1 + b^2, b^1 + b^3 + b^3) \end{aligned}$$

or, equivalently,

$$\begin{aligned} b^1 &= 1 \\ -b^1 + b^2 &= 2 \\ b^1 + b^2 + b^3 &= 3 \end{aligned}$$

- Unique solution is $(b^1, b^2, b^3) = (1, 3, -1)$

Change of Basis on Abstract Vector Space

- Consider two different bases of an n-dimensional vector space V,

$$E = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \text{ and } F = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}$$

- Since E is a basis, we can write each basis vector of F as a linear combination of the vectors in E

$$\begin{aligned} F &= \left[\; f_1 \;\middle|\; \cdots \;\middle|\; f_n \;\right] \\ &= \left[\; e_1 M_1^1 + \cdots + e_n M_1^n \;\middle|\; \cdots \;\middle|\; e_1 M_n^1 + \cdots + e_n M_n^n \;\right] \\ &= \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \begin{bmatrix} M_1^1 & \cdots & M_n^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_n^n \end{bmatrix} \\ &= EM, \end{aligned}$$

where M is a square matrix of scalars

Change of Coefficients

- Any vector v can be written as either a linear combination of the basis E,

$$v = e_1 a^1 + \cdots + e_n a^n = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} = Ea$$

or as a linear combination of the basis F,

$$v = f_1 b^1 + \cdots + f_n b^n = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} = Fb$$

- Since $F = EM$,

$$v = Fb = (EM)b = E(Mb)$$

- Therefore,

$$a = Mb \text{ and } b = M^{-1}a$$

Change of Basis Formula

- Let E and F be bases of V such that

$$F = EM,$$

- If $v = Ea = Fb$, then

$$a = Mb \text{ and } b = M^{-1}a$$

- The matrix that transforms old coefficients into new coefficients is the inverse of the matrix that transforms the old basis into the new basis
- Equivalently, the matrix that transforms the old basis into the new basis is the matrix that transforms the new coefficients into the old coefficients
- **WARNING:** This works only if you write a basis as a row matrix of vectors and the coefficients as a column matrix of scalars

Linear Functions

- If V is a vector space, then a function

$$\ell : V \to \mathbb{F}$$

is **linear**, if for any $v_1, v_2 \in V$

$$\ell(v_1 + v_2) = \ell(v_1) + \ell(v_2)$$

and for any $v \in V$ and $s \in \mathbb{F}$,

$$\ell(vs) = \ell(v)s$$

- Consequences:

$$\ell(0_V) = 0$$
$$\ell(-v) = -\ell(v)$$

Properties of Linear Functions

- If ℓ_1, ℓ_2 are linear functions, then so is $\ell_1 + \ell_2$
- If 0 is the zero function, it is linear and for any linear function ℓ,
$$\ell + 0 = \ell$$
- If $s \in \mathbb{F}$ and ℓ is a linear function, then the function $s\ell$, which is defined by
$$(s\ell)(v) = s(\ell(v)),$$
is also a linear function
- If we denote $-\ell = (-1)\ell$, then
$$\ell + (-\ell) = 0$$
- It is straightforward to verify that these operations satisfy the properties of vector addition and scalar multiplication
- It follows that the set of all linear functions on V, denoted V^*, is a vector space
- It is called the **dual vector space** of V

Linear Maps

- If V and W are vector spaces, then

$$L : V \to W$$

is a **linear map**, if for any $v, v_1, v_2 \in V$ and $s \in \mathbb{F}$,

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$
$$L(sv) = sL(v)$$

- Consequences:

$$L(0_V) = 0_W$$
$$L(-v) = -L(v)$$

Properties of Linear Maps

- If $K : U \to V$ and $L : V \to W$ are linear maps, then so is

$$L \circ K : U \to W$$

- If $L : V \to W$ is bijective, it is called a **linear isomorphism**
- If $L : V \to W$ is a linear isomorphism, then so is

$$L^{-1} : W \to V$$

n-Dimensional Vector Spaces are Isomorphic

- Let $\dim V = \dim W = m$
- Let $E = (e_1, \ldots, e_m)$ be a basis of V
- Let $F = (f_1, \ldots, f_m)$ be a basis of W
- The map

$$
\begin{aligned}
L_{E,F} &: V \to W \\
e_1 a^1 + \cdots + e_m a^m &\mapsto f_1 a^1 + \cdots + f_m a^m
\end{aligned}
$$

  is a linear isomorphism
- Given any basis $(e_1, \ldots, e_m)$ of V, there is a linear isomorphism

$$
\begin{aligned}
L_V &: \mathbb{F}^m \to V \\
(a^1, \ldots, a^m) &\mapsto e_1 a^1 + \cdots + e_m a^m
\end{aligned}
$$

Space of Linear Maps

- Let $\text{Hom}(V, W)$ denote the set of all linear maps with domain V and codomain W
- It is straightforward to check that if $L_1, L_2, L \in \text{Hom}(V, W)$ and $s \in \mathbb{F}$, then
$$L_1 + L_2, sL \in \text{Hom}(V, W)$$
are also linear maps from V to W
- It is also easily checked that these operations satisfy the properties of vector addition and scalar multiplication
- It follows that $\text{Hom}(V, W)$ is itself also a vector space
- Observe that $V^* = \text{Hom}(V, \mathbb{F})$

Endomorphisms and Automorphisms

- The space of **endomorphisms** of V is defined to be

$$\mathrm{End}(V) = \mathrm{Hom}(V, V)$$

- An endomorphism $L : V \to V$ is an **automorphism** if it is bijective
- The space of automorphisms of V is denoted $\mathrm{Aut}(V)$

Matrix as Linear Map

- Let $E = (e_1, \dots, e_m)$ be a basis of V
- Let $F = (f_1, \dots, f_n)$ be a basis of W
- For each $M \in \mathrm{gl}(n, m, \mathbb{F})$, let $L : V \to W$ be the linear map where
$$\forall\ 1 \le k \le m,\ \ L(e_k) = f_1 M^1_k + \cdots + f_n M^n_k$$
and therefore for any $v = e_1 a^1 + \cdots e_m a^m = Ea$,
$$\begin{aligned} L(v) &= L(e_1 a^1 + \cdots + e_m a^m) \\ &= L(e_1) a^1 + \cdots + L(e_m) a^m \\ &= (f_1 M^1_1 + \cdots + f_n M^n_1) a^1 + \cdots + (f_1 M^1_m + \cdots + f_n M^n_m) a^m \\ &= f_1 (M^1_1 a^1 + \cdots + M^1_m a^m) + \cdots f_n (M^n_1 a^1 + \cdots + M^n_m a^m) \\ &= f_1 (Ma)^1 + \cdots + f_n (Ma)^n \\ &= FMa \end{aligned}$$
- This defines a linear map $I_{E,F} : \mathrm{gl}(n, m, \mathbb{F}) \to \mathrm{Hom}(V, W)$

Linear Map as Matrix

- Let $E = (e_1, \ldots, e_m)$ be a basis of V
- Let $F = (f_1, \ldots, f_n)$ be a basis of W
- Let $L : V \to W$ be a linear map
- For each e_k, $1 \le k \le m$, there exists $(M_k^1, \ldots, M_k^n) \in \mathbb{F}^n$ such that $L(e_k) = f_1 M_k^1 + \cdots f_n M_k^n$
- Therefore, for any $v = e_1 a^1 + \cdots + e_m a^m \in V$,

$$\begin{aligned} L(v) &= L(e_1 a^1 + \cdots + e_m a^m) \\ &= L(e_1) a^1 + \cdots + L(e_m) e^m \\ &= (f_1 M_1^1 + \cdots f_n M_1^n) a^1 + \cdots + (f_1 M_m^1 + \cdots + f_n M_m^n) a^m \\ &= f_1 (M_1^1 a^1 + \cdots M_m^1 a^m) + \cdots + f_n (M_1^n a^1 + \cdots + M_m^n a^m) \\ &= f_1 (Ma)^1 + \cdots + f_n (Ma)^n \end{aligned}$$

- This defines a linear map $J_{E,F} : \mathrm{Hom}(V, W) \to \mathrm{gl}(n, m, \mathbb{F})$
- Since $J_{E,F} = I_{E,F}^{-1}$ and $I_{E,F} = J_{E,F}^{-1}$,

$$\dim \mathrm{Hom}(V, W) = \dim \mathrm{gl}(n, m, \mathbb{F}) = nm$$

Linear maps from $\mathbb{F}^m$ to $\mathbb{F}^n$ are Matrices

- Let $\mathrm{gl}(n, m, \mathbb{F})$ denote the vector space of n-by-m matrices with components in $\mathbb{F}$
  - $\dim \mathrm{gl}(n, m, \mathbb{F}) = nm$
  - Let $\mathrm{gl}(n, \mathbb{F}) = \mathrm{gl}(n, n, \mathbb{F})$
  - Let $\mathrm{gl}(n) = \mathrm{gl}(n, \mathbb{R})$
- If E is the standard basis of $\mathbb{F}^m$ and F is the standard basis of $\mathbb{F}^n$, then $J_{E,F}$ is a natural isomorphism

$$\mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^n) = \mathrm{gl}(n, m, \mathbb{F})$$

Concrete to Abstract Notation

$$
\begin{aligned}
L(v) &= L(e_1 a^1 + \cdots + e_m a^m) = L\left(\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}\right) \\
&= L\left(\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}\right) \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = \begin{bmatrix} L(e_1) & \cdots & L(e_m) \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} \\
&= \begin{bmatrix} f_1 M_1^1 + \cdots + f_n M_1^n & \cdots & f_1 M_n^1 + \cdots + f_n M_n^n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} \\
&= \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = FMa
\end{aligned}
$$

Change of Basis Formula for Linear Map

- Let $E = (e_1, \ldots, e_m)$ be a basis of V
- Let $F = (f_1, \ldots, f_m)$ be another basis of V
- There is a matrix B such that $F = EB$, i.e.,

$$f_k = e_j B^j_k$$

- Consider a linear map $L : V \to V$
- There is a matrix M such that

$$L(e_k) = e_j M^j_k, \text{ i.e., } L(E) = EM$$

  and a matrix N such that

$$L(f_k) = f_j N^j_k, \text{ i.e., } L(F) = FN$$

- It follows that

$$FN = L(F) = L(EB) = L(E)B = EMB = FB^{-1}MB$$

  and therefore $N = B^{-1}MB$

Change of Basis Formula for a Matrix

- Let $E = (e_1, \dots, e_m)$ be the standard basis and $F = (f_1, \dots, f_m)$ be another basis of $\mathbb{F}^m$
- Let M be an m-by-m matrix and $L : \mathbb{F}^m \to \mathbb{F}^m$ be the linear map where
$$L(E) = FM$$
- There also exists a matrix N such that $L(F) = FN$
- The change of basis matrix from E to F is an invertible matrix B such that
$$F = EB, \text{ i.e., } f_k = e_j B^j_k$$
It also follows that $E = FB^{-1}$
- It follows that
$$FN = L(F) = L(EB) = L(E)B = EMB = FB^{-1}MB$$
and therefore the change of basis formula for linear map L is
$$N = B^{-1}MB$$

Composition is Matrix Multiplication

- Consider vector spaces U, V, W and linear maps

$$K : U \to V,\ \ L : V \to W$$

- Let $(e_1, \ldots, e_k)$ be a basis of U
- Let $(f_1, \ldots, f_m)$ be a basis of V
- Let $(g_1, \ldots, g_n)$ be a basis of W
- There is an m-by-k matrix M such that

$$K(e_j) = f_p M^p_j,\ \ 1 \leq j \leq k$$

- There is an n-by-m matrix N such that

$$L(f_p) = g_a N^a_p,\ \ 1 \leq p \leq m$$

- There is an n-by-k matrix P such that

$$(L \circ K)(e_j) = g_a P^a_j,\ \ 1 \leq j \leq k$$

- On the other hand,

$$(L \circ K)(e_j) = L(K(e_j)) = L(f_p M^p_j) = L(f_p)M^p_j = g_a N^a_p M^p_j$$

- Therefore, $P^a_j = N^a_p M^p_j$.