MA-GY 7043: Linear Algebra II

Determinants

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Outline I

Alternating Multilinear Function on Permutation of Basis

- Let $T \in \Lambda^n V^*$
- Let $(e_1, \ldots, e_n)$ be a basis of V
- For each $\phi \in S_n$,

$$T(e_{\phi(1)}, \ldots, e_{\phi(n)}) = \epsilon(\phi) T(e_1, \ldots, e_n)$$

- If $\phi \in \mathrm{End}(n) \backslash S_n$, then there exists $1 \le j, k \le n$ such that $\phi(j) = \phi(k)$ and therefore

$$T(e_{\phi(1)}, \ldots, e_{\phi(n)}) = 0$$

- It follows that for any $\phi \in \mathrm{End}(n)$,

$$T(e_{\phi(1)}, \ldots, e_{\phi(n)}) = \epsilon(\phi) T(e_1, \ldots, e_n)$$

Alternating Multilinear Function With Respect to Basis

- If $(e_1, e_2, \dots, e_n)$ is a basis of V and $v_k = e_j a_k^j,\ 1 \le k \le n$,

$$
\begin{aligned}
T(v_1, \dots, v_n) &= T(e_{j_1} a_1^{j_1}, \dots, e_{j_n} a_n^{j_n}) \\
&= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} T(e_{j_1}, \cdots, e_{j_n}) a_1^{j_1} \cdots a_n^{j_n} \\
&= \sum_{\phi \in \mathrm{End}(n)} T(e_{\phi(1)}, \cdots, e_{\phi(n)}) a_1^{\phi(1)} \cdots a_n^{\phi(n)} \\
&= \sum_{\phi \in \mathrm{End}(n)} \epsilon(\phi) T(e_1, \cdots, e_n) a_1^{\phi(1)} \cdots a_n^{\phi(n)} \\
&= T(e_1, \dots, e_n) \sum_{\phi \in \mathrm{End}(n)} \epsilon(\phi) a_{\phi(1)}^1 \cdots a_{\phi(n)}^n \\
&= T(e_1, \dots, e_n) \sum_{\phi \in S_n} \epsilon(\phi) a_{\phi(1)}^1 \cdots a_{\phi(n)}^n
\end{aligned}
$$

Determinant of a Matrix

- Define the determinant of an n-by-n matrix M to be

$$\begin{aligned}\det(M) &= \sum_{\phi \in S_n} \epsilon(\phi) M^1_{\phi(1)} \cdots M^n_{\phi(n)} \\ &= \sum_{\phi \in \mathrm{End}(n)} \epsilon(\phi) M^1_{\phi(1)} \cdots M^n_{\phi(n)}\end{aligned}$$

Basic Properties of Alternating Multilinear Functions

- Given a basis $E = (e_1, \ldots, e_n)$, any $T \in \Lambda^n V^*$ is uniquely determined by $T(e_1, \ldots, e_n)$, because if

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = EM,$$

then

$$T(v_1, \ldots, v_n) = T(e_1, \ldots, e_n)\det(M)$$

- In particular, $T \neq 0$ if and only if $T(e_1, \ldots, e_n) \neq 0$.

Space of Alternating Multilinear Functions

- If $S_1, S_2 \in \Lambda^n V^*$ and $a_1, a_2 \in \mathbb{F}$, then $a_1 S_1 + a_2 S_2 \in \Lambda^n V^*$
- Therefore, $\Lambda^n V^*$ is a vector space
- Let $T \in \Lambda^n V^* \backslash \{0\}$ and $(e_1, \ldots, e_n)$ be a basis of V
- For any $v_1, \ldots, v_n \in V$, Let M be the matrix such that

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = EM$$

- It follows that

$$\begin{aligned} S(v_1, \ldots, v_n) &= S(e_1, \ldots, e_n) \det(M) \\ &= cT(e_1, \ldots, e_n) \det(M) \\ &= cT(v_1, \ldots, v_n) \end{aligned}$$

- Therefore, $S = cT$
- It follows that $\dim(\Lambda^n V^*) = 1$ and any $T \in \Lambda^n V^* \backslash \{0\}$ is a basis

Pullback of Oriented Volume Function by Linear Map

- Let V and W be n-dimensional vector spaces
- Let $L : V \to W$ be a linear map
- Let $T \in \Lambda^n W^*$ be an oriented volume function
- Use L and T to define $S \in \Lambda^n V^*$ as follows:

$$S(v_1, \ldots, v_n) = T(L(v_1), \ldots, L(v_n))$$

- S is called the **pullback** of T by the linear map L and denoted $L^* T$
- The pullback is the linear map

$$L^* : \Lambda^n W^* \to \Lambda^n V^*,$$

where for any $T \in \Lambda^n V^*$ and $v_1, \ldots, v_n \in V$,

$$L^* T(v_1, \ldots, v_n) = T(L(v_1), \ldots, L(v_n))$$

Composition of Pullbacks

- Let V_0, V_1, V_2 be n-dimensional vector spaces and
$$L_1 : V_0 \to V_1 \text{ and } L_2 : V_1 \to V_2$$
be linear maps
- Their pullbacks are linear maps
$$L_1^* : \Lambda^n V_1^* \to \Lambda^n V_0^* \text{ and } L_2^* : \Lambda^n V_2^* \to \Lambda^n V_1^*$$
- The pullback of the their composition is the linear map
$$(L_2 \circ L_1)^* : \Lambda^n V_2^* \to \Lambda^n V_0^*$$
- Then for each $T \in \Lambda^n V_2^*$ and $v_1, \dots, v_n \in V_0$,
$$\begin{aligned}(L_2 \circ L_1)^*(T)(v_1, \dots, v_n) &= T(L_2 \circ L_1(v_1), \dots, L_2 \circ L_1(v_n)) \\ &= T(L_2(L_1(v_1)), \dots, L_2(L_1(v_n))) \\ &= L_2^*(T)(L_1(v_1), \dots, L_1(v_n)) \\ &= L_1^*(L_2^*(T))(v_1, \dots, v_n) \\ &= (L_1^* \circ L_2^*)(T)(v_1, \dots, v_n)\end{aligned}$$
- Therefore, $(L_2 \circ L_1)^* = L_1^* \circ L_2^*$

Determinant of a Linear Transformation

- Let $L : V \to V$ be a linear transformation
- Let $T \in \Lambda^n V^*$
- Therefore, the pullback is a linear map

$$L^* : \Lambda^n V^* \to \Lambda^n V^*$$

- Since $\dim(\Lambda^n V^*) = 1$, it follows that for any $T \in \Lambda^n V^* \backslash \{0\}$, there exists $c \in \mathbb{F}$ such that

$$L^*(T) = cT$$

- c is called the **determinant** of L, denoted $\det(L)$, and defines a (nonlinear) function

$$\det : \mathrm{End}(V) \to \mathbb{F}$$

- If $I : V \to V$ is the identity map, then $\det(I) = 1$

Example (Part 1)

- Consider the matrix

$$M = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & -1 \\ -2 & 0 & 0 \end{bmatrix}$$

- Let $V = \mathbb{F}^3$, $E = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{F}^3$, and $L : V \to V$ be the linear map given by

$$\begin{aligned} L(E) &= EM \\ &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & -1 \\ -2 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e_2 - 2e_3 & 3e_1 & e_1 - e_2 \end{bmatrix} \end{aligned}$$

- I.e.,

$$\begin{aligned} L(e_1) &= e_2 - 2e_3 \\ L(e_2) &= 3e_1 \\ L(e_3) &= e_1 - e_2 \end{aligned}$$

Example (Part 2)

- Let $D \in \Lambda^3 V^*$ satisfy $D(e_1, e_2, e_3) = 1$
- The determinant of L can be calculated as follows:

$$\begin{aligned}
\det(L) &= (L^* D)(e_1, e_2, e_3) \\
&= D(L(e_1), D(L(e_2)), D(L(e_3)) \\
&= D(e_2 - 2e_3, 3e_1, e_1 - e_2) \\
&= D(e_2 - 2e_3, 3e_1, e_1) + D(e_2 - 2e_3, 3e_1, -e_2) \\
&= 0 + D(e_2, 3e_1, -e_2) + D(-2e_3, 3e_1, -e_2) \\
&= (-2)(3)(-1) D(e_3, e_1, e_2) \\
&= 6 D(e_1, e_2, e_3) \\
&= 6
\end{aligned}$$

Determinant of Composition of Linear Transformations

- If $L_1 : V \to V$ and $L_2 : V \to V$ are linear transformations, then

$$\det(L_2 \circ L_1) = \det(L_1)\det(L_2)$$

- L is invertible if and only if $\det(L) \neq 0$
- If L is invertible, then

$$\det(L^{-1}) = \frac{1}{\det(L)}$$

Determinant of a Matrix Product

- Let $E = (e_1, \ldots, e_n)$ be the standard basis of $\mathbb{F}^n$
- Recall that a matrix $M \in \mathrm{gl}(n, \mathbb{F})$ defines a linear map

  $$L_M : \mathbb{F}^n \to \mathbb{F}^n,$$

  where if E is the standard basis, then $L_M(E) = EM$ and therefore

  $$\det(L_M) = \det(M)$$

- Recall that if $M_1, M_2 \in \mathrm{gl}(n, \mathbb{F})$ and $v = Ea \in \mathbb{F}^n$, then

  $$\begin{aligned} L_{M_2} \circ L_{M_1}(Ea) &= L_{M_2}(L_{M_1}(Ea)) \\ &= L_{M_2}(EM_1 a) \\ &= EM_2 M_1 a \end{aligned}$$

- Therefore,

  $$\begin{aligned} \det(M_2)\det(M_1) &= \det(L_{M_2})\det(L_{M_1}) \\ &= \det(L_{M_2} \circ L_{M_1}) \\ &= \det(M_2 M_1) \end{aligned}$$

Transpose of a Matrix

- Given a matrix $M \in \mathrm{gl}(n, m, \mathbb{F})$, its transpose is the matrix $M^T \in \mathrm{gl}(m, n, \mathbb{F})$ that switches the rows and columns
- In other words,
$$(M^T)^j_k = M^k_j$$
- Or
$$\begin{bmatrix} M^1_1 & \cdots & M^1_m \\ \vdots & & \vdots \\ M^n_1 & \cdots & M^n_m \end{bmatrix}^T = \begin{bmatrix} M^1_1 & \cdots & M^n_1 \\ \vdots & & \vdots \\ M^1_m & \cdots & M^n_m \end{bmatrix}$$
- If $M \in \mathcal{M}_{n\times m}$, then $M^T \in \mathrm{gl}(m, n, \mathbb{F})$
- For any $A \in \mathcal{M}_{k\times m}$ and $B \in \mathrm{gl}(m, n, \mathbb{F})$, then $AB \in \mathcal{M}_{k\times n}$ and
$$(AB)^T = B^T A^T \in \mathcal{M}_{n\times k}$$

Determinant of Matrix Equals Determinant of Its Transpose

- Lemma: Given any square matrix M,

$$\det M^T = \det M$$

- Proof: Use the formula for the determinant

$$\begin{aligned}
\det M &= \sum_{\sigma \in S_n} \epsilon(\sigma) M_1^{\sigma(1)} \cdots M_n^{\sigma(n)} \\
&= \sum_{\sigma \in S_n} \epsilon(\sigma) M^1_{\sigma^{-1}(1)} \cdots M^n_{\sigma^{-1}(n)} \\
&= \sum_{\sigma \in S_n} \epsilon(\sigma) M^1_{\sigma(1)} \cdots M^n_{\sigma(n)} \\
&= \det M^T
\end{aligned}$$