MA-GY 7043: Linear Algebra II

Determinant of Diagonal and Triangular Matrices
Eigenvalues and Eigenvectors

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Outline I

Diagonal Matrix

- An n-by-n matrix M is **diagonal** if the only nonzero entries are along the diagonal

$$M = \begin{bmatrix} M_1^1 & 0 & \cdots & 0 & 0 \\ 0 & M_2^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{n-1}^{n-1} & 0 \\ 0 & 0 & \cdots & 0 & M_n^n \end{bmatrix}$$

Diagonal Basis of a Linear Transformation

- Given a linear transformation $L : V \to V$, a basis $E = (e_1, \dots, e_n)$is **diagonal** if for each $k \in \{1, \dots, n\}$, there exists $d_k \in \mathbb{F}$ such that

$$L(e_k) = e_k d_k,$$

- A basis E is diagonal for L if and only if $L(E) = EM$, where M is a diagonal matrix
- A linear transformation L is **diagonalizable** if there exists a diagonal basis for L
- Not every linear transformation is diagonalizable

Determinant of Diagonalizable Linear Transformation

- Let $L : V \to V$ be a linear transformation with a diagonal basis $E = (e_1, \ldots, e_n)$, i.e.,

$$L(e_k) = e_k d_k$$

- Let $D_E \in \Lambda^n V^*$ satisfy $D(e_1, \ldots, e_n) = 1$
- The determinant of L is

$$\begin{aligned} \det(L) &= D_E(L(e_1), \ldots, L(e_n)) \\ &= D_E(e_1 d_1, \ldots, e_n d_n) \\ &= d_1 \cdots d_n D(e_1, \ldots, e_n) \\ &= d_1 \cdots d_n \end{aligned}$$

- It follows that the determinant of a diagonal matrix M is the product of the diagonal elements,

$$\det(M) = M_1^1 \cdots M_n^n$$

Triangular Matrix

- An n-by-n matrix M is **upper triangular** if it is of the form

$$M = \begin{bmatrix} M^1_1 & M^1_2 & \cdots & M^1_{n-1} & M^1_n \\ 0 & M^2_2 & \cdots & M^2_{n-1} & M^2_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M^{n-1}_{n-1} & M^{n-1}_n \\ 0 & 0 & \cdots & 0 & M^n_n \end{bmatrix}$$

- An n-by-n matrix M is **lower triangular** if it is of the form

$$M = \begin{bmatrix} M^1_1 & 0 & \cdots & 0 & 0 \\ M^2_1 & M^2_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M^{n-1}_1 & M^{n-1}_2 & \cdots & M^{n-1}_{n-1} & 0 \\ M^n_1 & M^n_2 & \cdots & M^n_{n-1} & M^n_n \end{bmatrix}$$

- A matrix M is **triangular** if it is upper or lower triangular

Triangular Basis of a Linear Transformation

- A basis E is **triangular** for a linear transformation $L : V \to V$ if $L(E) = EM$, where M is a triangular matrix
- If M is upper triangular, then

$$
\begin{aligned}
L(e_1) &= e_1 M_1^1 \\
L(e_2) &= e_1 M_2^1 + e_2 M_2^2 \\
\vdots \quad & \quad \vdots \\
L(e_{n-1}) &= e_1 M_{n-1}^1 + e_2 M_{n-1}^2 + \cdots + e_{n-1} M_{n-1}^{n-1} \\
L(e_n) &= e_1 M_n^1 + e_2 M_n^2 + \cdots + e_{n-1} M_n^{n-1} + e_n M_n^n
\end{aligned}
$$

- An analogous set of equations hold if M is lower triangular

Determinant of a Triangular Matrix

- Let $E = (e_1, \ldots, e_n)$ be a triangular basis for $L : V \to$ such that

$$L(E) = EM,$$

  where M is upper triangular
- Let $D_E \in \Lambda^n V^*$ be as before
- Then

$$\begin{aligned}
\det(L) &= D_E(L(e_1), L(e_2), \cdots, L(e_n)) \\
&= D_E(e_1 M_1^1, e_1 M_2^1 + e_2 M_2^2, \cdots, e_1 M_n^1 + \cdots + e_n M_n^n) \\
&= M_1^1 D_E(e_1, e_1 M_2^1 + e_2 M_2^2, \cdots, e_1 M_n^1 + \cdots + e_n M_n^n) \\
&= M_1^1 M_2^2 D_E(e_1, e_2, e_1 M_3^1 + e_2 M_3^2 + e_3 M_3^3, \ldots, e_n M_n^n) \\
&= M_1^1 M_2^2 \cdots M_n^n D_E(e_1, \ldots, e_n) \\
&= M_1^1 M_2^2 \cdots M_n^n
\end{aligned}$$

- It follows that if M is upper triangular, its determinant is the product of its diagonal elements
- A similar calculation shows the same for a lower triangular matrix

Eigenvalues and Eigenvectors of a Linear Transformation

- Consider a linear transformation $L : V \to V$
- A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of L if there is a nonzero vector $v \in V$ such that any of the following equivalent statements hold:

$$L(v) = \lambda v \iff (L - \lambda I)v = 0$$
$$\iff v \in \ker(L - \lambda I)$$

- The vector v is called an **eigenvector** for the eigenvalue λ
- $\lambda \in \mathbb{F}$ is an eigenvalue of L if and only if the following equivalent statements hold:

$$\dim(\ker(L - \lambda I)) > 0 \iff \det(L - \lambda I) = 0$$

- The **eigenspace** for an eigenvalue λ of L is

$$E_\lambda(L) = \ker(L - \lambda I) = \{v \in V \ : \ L(v) = \lambda v\}$$

- $E_\lambda(L)$ is a linear subspace of V
- The **geometric multiplicity** of the eigenvalue λ is $\dim(E_\lambda(L))$

Eigenvalues and Eigenvectors of a Square Matrix

- A scalar $\lambda \in \mathbb{F}$ is an **eigenvalue** of a matrix $M \in \mathrm{gl}(n, \mathbb{F})$ if there is a nonzero vector $v \in \mathbb{F}^n$ such that any of the following equivalent statements hold:

$$\begin{aligned} Mv = \lambda v &\iff (M - \lambda I)v = 0 \\ &\iff v \in \ker(M - \lambda I) \end{aligned}$$

- The vector v is called an **eigenvector** for the eigenvalue λ
- $\lambda \in \mathbb{F}$ is an eigenvalue of M if and only if the following equivalent statements hold:

$$\dim(\ker(M - \lambda I)) > 0 \iff \det(M - \lambda I) = 0$$

- The **eigenspace** for an eigenvalue λ is the subspace

$$E_\lambda(M) = \ker(M - \lambda I) = \{v \in \mathbb{V} : \; Mv = \lambda v\}$$

- The **geometric multiplicity** of the eigenvalue λ is $\dim(E_\lambda(M))$

Eigenvalues of Linear Transformation Versus Matrix

- Let $L : V \to V$ be a linear transformation
- Let $E = (e_1, \ldots, e_n)$ be a basis of V and M be the matrix such that
$$L(E) = EM$$
- If $v = Ea$ is an eigenvector of L for an eigenvalue λ, then
$$\lambda v = L(v) = L(Ea) = L(E)a = EMa$$
and therefore
$$\lambda Ea = EMa$$
- It follows that
$$Ma = \lambda a,$$
- Therefore, $v = Ea$ is an eigenvector of L for the eigenvalue λ if and only if $a \in \mathbb{F}^n$ is an eigenvector of M for the eigenvalue λ

Linear Transformation With Respect To Different Bases

- Let E and F be bases of V
- There exists a matrix S such that $f_k = e_j S^j_k$, i.e.,

$$F = ES \text{ and } E = FS^{-1}$$

- Given a map $L : V \to V$, there are matrices M and N such that

$$L(E) = EM \text{ and } L(F) = FN$$

- On the other hand,

$$FN = L(F) = L(ES) = L(E)S = EMS = FS^{-1}MS$$

and therefore,

$$N = S^{-1}MS$$

- If $v = Ea = Fb$, then

$$L(v) = EMa = FNb = ESNb = ESS^{-1}MSb = EMSb$$

- Therefore,

$$a = Sb \text{ and } b = S^{-1}a$$

Eigenvalues, and Eigenvectors of Similar Matrices

- Two matrices M and N are called **similar** if there is an invertible matrix S such that
$$N = S^{-1}MS$$
or, equivalently,
$$M = SNS^{-1}$$
- If M and N are similar, then $\det M = \det N$
- M and N have the same eigenvalues, because if a is an eigenvector of M for the eigenvalue λ and $b = S^{-1}a$, then
$$Nb = S^{-1}MSb = S^{-1}Ma = S^{-1}(\lambda a) = \lambda S^{-1}a = \lambda b$$

Characteristic Polynomial of a Matrix

- Let δ^j_k be the element in the j-th row and k-column of the identity matrix, i.e.,

$$\delta^j_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

- Observe that the function $p_M : \mathbb{F} \to \mathbb{F}$ given by

$$\begin{aligned} p_M(x) &= \det(M - xI) \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma)(M - xI)^{\sigma(1)}_1 \cdots (M - xI)^{\sigma(n)}_n \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma)(M^{\sigma(1)}_1 - x\delta^{\sigma(1)}_1) \cdots (M^{\sigma(n)}_n - x\delta^{\sigma(n)}_n) \end{aligned}$$

  is a polynomial in x of degree n
- p_M is the **characteristic polynomial** of M
- x is a root of p_M if and only if it is an eigenvalue for M

Characteristic Polynomial of a Linear Transformation

- Let $L : V \to V$ be a linear transformation
- Define $p_L : \mathbb{F} \to \mathbb{F}$ by

$$p_L(x) = \det(L - xI)$$

- If E is a basis and $L(E) = EM$, then

$$(L - xI)(E) = E(M - xI)$$

and therefore

$$p_L(x) = \det(L - xI) = \det(M - xI) = p_M(x)$$

- It follows that p_L is a polynomial of degree n

Similar Matrices Have the Same Characteristic Polynomial

- **Proof 1:** If $L(E) = EM$ and $L(F) = FN$, then

$$p_M(x) = p_L(x) = p_N(x)$$

- **Proof 2:** If $M = SNS^{-1}$, then

$$M - xI = S(N - xI)S^{-1}$$

and therefore

$$\begin{aligned} p_M(x) &= \det(M - xI) \\ &= \det(S(N - xI)S^{-1}) \\ &= \det(S)\det(N - xI)\det(S^{-1}) \\ &= \det(N - xI) = p_N(x) \end{aligned}$$

Examples

- Let
$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
- $Zv = 0v$ for any $v \in \mathbb{R}^2$ and therefore 0 is the only eigenvalue
- Any nonzero vector $v \in \mathbb{R}^2$ is an eigenvector
- The characteristic polynomial is
$$p_Z(x) = \det(Z - xI) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = x^2$$

Examples

- If $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then $D \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} av^1 \\ bv^2 \end{bmatrix}$
- If $x = a = b$, then the only eigenvalue is x
  - Every $v \in \mathbb{R}^2$ is an eigenvector
- If $a \neq b$, then the only eigenvalues are a and b
  - The eigenvectors for the eigenvalue a are

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x \in \mathbb{F} \backslash \{0\}$$

  - The eigenvectors for the eigenvalue b are

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$p_D(x) = \det(D - xI) = x \begin{bmatrix} a - x & 0 \\ 0 & b - x \end{bmatrix} = (a - x)(b - x)$$

Examples

- If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^2 \\ v^1 \end{bmatrix}$
- The only eigenvalues are $1, -1$
- The eigenvectors for the eigenvalue 1 are

$$\begin{bmatrix} x \\ x \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue -1 are

$$\begin{bmatrix} x \\ -x \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$p_A(x) = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}\right) = \det\left(\begin{bmatrix} -x & 1 \\ 1 & x \end{bmatrix}\right) = 1 - x^2$$

Examples

- If $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $B \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} -v^2 \\ v^1 \end{bmatrix}$
- There are no real eigenvalues
- The complex eigenvalues are $i, -i$
- The eigenvectors for the eigenvalue i are

$$\begin{bmatrix} ix \\ -x \end{bmatrix} = x \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue $-i$ are

$$\begin{bmatrix} x \\ ix \end{bmatrix} = x \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$\begin{aligned} p_B(x) &= \det(B - xI) \\ &= \det \left(\begin{bmatrix} -x & -1 \\ 1 & -x \end{bmatrix} \right) \\ &= 1 + x^2 \end{aligned}$$

Complex Versus Real Eigenvalues

- If an $n - by - n$ matrix contains only real entries, it can have anywhere from 0 to n eigenvalues
- A polynomial with complex coefficients

$$p(x) = a_0 + a_1 x + \cdots a_n x^n,$$

where $a_n \neq 0$ with complex coefficients can always be factored into n linear factors

$$p(x) = a_n(r_1 - x) \cdots (r_n - x)$$

- A complex matrix A always has anywhere from 1 to n eigenvalues, where an eigenvalue might appear more than once in the factorization of p_A
- The **algebraic multiplicity** of an eigenvalue λ is the number of linear factors equal to $(\lambda - x)$ in p_A

Examples

- Let $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- The eigenvalues of D are $-2, 3$
- The characteristic polynomial of D is

$$p_D(\lambda) = (x-3)(x+2)(x-3) = (x-3)^2(x+2)$$

- The eigenvalue 3 has multiplicity 2, and the eigenvalue 2 has multiplicity 1
- The eigenvectors for the eigenvalue -2 are

$$\begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue 3 are

$$\begin{bmatrix} x^1 \\ 0 \\ x^2 \end{bmatrix} = x^1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

Examples

- Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- The characteristic polynomial of M is

$$p_M(\lambda) = \det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2$$

- The only eigenvalue is 1 with multiplicity 2
- Since

$$M \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^1 + v^2 \end{bmatrix},$$

the eigenvectors of the eigenvalue 1 are

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalues and Eigenvectors of a Diagonalizable Linear Transformation

- Let $E = (e_1, \ldots, e_n)$ be a diagonal basis of $L : V \to V$
- For each e_k, there is a $d_k \in \mathbb{F}$ such that

$$L(e_k) = d_k e_k$$

- Since $e_k \neq 0$, it follows that each e_k is an eigenvector for the eigenvalue d_k
- Conversely, if there exists a basis of eigenvectors, $(e_1, \ldots, e_n)$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then $(e_1, \ldots, e_n)$ is a diagonal basis for L

Characteristic Polynomial and Determinant of Triangular Matrix

- The matrix $M - xI$ is also triangular, and its diagonal elements are
$$M_1^1 - x, \dots, M_n^n - x$$
- The characteristic polynomial of a triangular matrix M is
$$p_M(\lambda) = (M_1^1 - \lambda I) \cdots (M_n^n - \lambda I)$$
- The eigenvalues of M are the diagonal elements of M
- The eigenvectors have no simple formula