MA-GY 7043: Linear Algebra II

Diagonalizable Transformations and Matrices
Inner Product Spaces

Deane Yang

Courant Institute of Mathematical Sciences
New York University

Feburary 23, 2026

Outline I

Diagonalizable Transformations and Matrices

- Recall that a linear transformation $L : V \to V$ is **diagonalizable** if there exists a basis $(v_1, \dots, v_n)$ and scalars $\lambda_1, \dots, \lambda_n$ such that for each $k \in \{1, \dots, n\}$

$$L(v_k) = \lambda_k v_k,$$

  i.e., V has a basis of eigenvectors of L
- A matrix is **diagonal** if the standard basis vectors of $\mathbb{F}^n$ are eigenvectors
- A matrix M is **diagonalizable** if it is similar to a diagonal matrix
- Equivalently, there exists an invertible matrix S such that the matrix

$$D = S^{-1}MS$$

  is diagonal

Diagonalization of a Matrix

- Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{F}^n$
- Let M be a diagonalizable matrix
- Let $(v_1, \ldots, v_n)$ be a basis of eigenvectors of M with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$
- Let S be the matrix such that for each $k \in \{1, \ldots, n\}$,

$$Se_k = v_k,$$

  i.e., the columns of S are the eigenvectors $v_1, v_2, \ldots, v_n$
- If $D = S^{-1}MS$, then for each standard basis vector e_k,

$$De_k = S^{-1}MSe_k = S^{-1}Mv_k = S^{-1}(\lambda_k v_k) = \lambda_k S^{-1} v_k = \lambda_k e_k$$

- Therefore, the matrix $D = S^{-1}MS$ is diagonal, where the elements along the diagonal are $(\lambda_1, \ldots, \lambda_n)$

Diagonalizable Matrix with Given Eigenvectors and Eigenvalues

- Conversely, given
  - $(c_1, \ldots, c_n)$ is a basis of $\mathbb{F}^n$
  - $d_1, \ldots, d_n \in \mathbb{F}$,

  let
  - D be the diagonal matrix whose elements along the diagonal are $(d_1, \ldots, d_n)$
  - S be the matrix whose columns are $(c_1, \ldots, c_n)$,
  - $M = SDS^{-1}$
- For each $k \in \{1, \ldots, n\}$,

$$Mc_k = SDS^{-1}c_k = SDe_k = Sd_k e_k = d_k Se_k = d_k c_k$$

- It follows that $(c_1, \ldots, c_n)$ is a basis of eigenvectors of M, and the corresponding eigenvalues are $(d_1, \ldots, d_n)$

Linear Transformation With Distinct Eigenvalues is Diagonalizable

- **Theorem.** *If* $\dim(V) = n$ *and* $L : V \to V$ *be a linear transformation with* n *distinct eigenvalues* $\lambda_1, \ldots, \lambda_n$*, i.e.,*

$$j \neq k \implies \lambda_j \neq \lambda_k,$$

then L *is diagonalizable*

Proof

- Let $v_1, \dots, v_n$ be eigenvectors for the eigenvalues $\lambda_1, \dots, \lambda_n$ respectively
- Proof is by mathematical induction
- We want to prove the following: *For each $j \in \{1, \dots, n\}$, the set $\{v_1, \dots, v_j\}$ is linearly independent*
- Since $v_1 \neq 0$, this statement is true when $j = 1$
- We want to show that if the statement is true for $j = k < n$, then it is also true for $j = k + 1$
- If $v_1 a^1 + \cdots + v_k a^k + v_{k+1} a^{k+1} = 0$, then

$$
\begin{aligned}
0 &= (L - \lambda_{k+1} I)(v_1 a^1 + \cdots + v_k a^k + v_{k+1} a^{k+1}) \\
&= (L(v_1) - \lambda_k v_1) a^1 + \cdots + (L(v_{k+1}) - \lambda_{k+1} v_{k+1}) a^{k+1} \\
&= v_1 a^1 (\lambda_1 - \lambda_{k+1}) + \cdots + v_k a^k (\lambda_k - \lambda_{k+1}) \\
&\quad + v_{k+1} a^{k+1} (\lambda_{k+1} - \lambda_{k+1}) \\
&= v_1 a^1 (\lambda_1 - \lambda_{k+1}) + \cdots + v_k a^k (\lambda_k - \lambda_{k+1})
\end{aligned}
$$

Linear Transformation With Distinct Eigenvalues (Part 2)

- Since $v_1, \ldots, v_k$ are linearly independent, it follows that

$$a^1(\lambda_1 - \lambda_{k+1}) = \cdots = a^{k-1}(\lambda_k - \lambda_{k+1}) = 0$$

- Since $\lambda_j - \lambda_{k+1} \neq 0$ for each $j \in \{1, \ldots, k\}$, it follows that

$$a^1 = \cdots = a^k = 0$$

- This implies that

$$v_{k+1}a^{k+1} = -(v_1a^1 + \cdots + v_ka^k) = 0$$

- Since $v_{k+1} \neq 0$, it follows that $a^{k+1} = 0$
- Therefore, $a_1 = \cdots = a_{k+1} = 0$
- This implies that $\{v_1, \ldots, v_{k+1}\}$ is linearly independent
- By induction, it follows that $v_1, \ldots, v_n$ is basis of V
- Therefore, L is diagonalizable

Direct Sum of Subspaces

- Let $V_1, \ldots, V_k$ be subspaces of V
- $\{V_1, \ldots, V_k\}$ is a **linearly independent** set of subspaces if for any nonzero vectors

$$v_1 \in V_1,\ v_2 \in V_2, \ldots, v_k \in V_k,$$

  the set $\{v_1, \ldots, v_k\}$ is linearly independent
- Equivalently, $\{V_1, \ldots, V_k\}$ is linearly independent if for any vectors $v_1 \in V_1, \ldots, v_k \in V_k$,

$$v_1 + v_2 + \cdots + v_k = 0 \implies v_1 = v_2 = \cdots = v_k = 0$$

- Equivalently, $\{V_1, \ldots, V_k\}$ is linearly independent if for any $v_1, w_1 \in V_1, \ldots, v_k, w_k \in V_k$,

$$v_1 + v_2 + \cdots + v_k = w_1 + w_2 + \cdots + w_k \implies v_1 = w_1, \ldots, v_k = w_k$$

- If $\{V_1, V_2, \ldots, V_k\}$ is linearly independent, then their **direct sum** is defined to be

$$V_1 \oplus V_2 \oplus \cdots \oplus V_k = \operatorname{span}(V_1 \cup V_2 \cup \cdots \cup V_k)$$

Examples

- $\{S_1, S_2\}$, where $S_1, S_2 \subset \mathbb{F}^3$ are given by

$$S_1 = \text{span}(e_1)$$
$$S_2 = \text{span}(e_2),$$

  is linearly independent
- If $\{v_1, \dots, v_k\}$ is linearly independent and for each $j \in \{1, \dots, k\}$,

$$V_j = \text{span}(v_j),$$

  then $\{V_1, \dots, V_k\}$ is a linearly independent set of subspaces
- If (e_1, e_2, e_3, e_4) is a basis of V and

$$S = \text{span}(e_1, e_2, e_3), \;\; T = \text{span}(e_4),$$

  then $V = S \oplus T$

Eigenspaces of Distinct Eigenvalues are Linearly Independent (Part 1)

- We want to prove: *If* $\lambda_1, \ldots, \lambda_k$ *are distinct eigenvalues of* $L : V \to V$*, then their eigenspaces* $E_{\lambda_1}, \ldots, E_{\lambda_k}$ *are linearly independent*
- Prove by induction that for any $1 \leq j \leq k$,

$$v_1 + \cdots + v_j = 0 \implies v_1 = \cdots = v_j = 0$$

- This holds for $j = 1$
- Inductive step: Assume that it holds for $j < k$ and prove it also holds holds for $j + 1$

Eigenspaces of Distinct Eigenvalues are Linearly Independent (Part 2)

- Suppose $v_1 \in E_{\lambda_1}, \ldots, v_{j+1} \in E_{\lambda_{j+1}}$ satisfy

$$v_1 + \cdots + v_{j+1} = 0 \tag{1}$$

- Since $(L - \lambda_{j+1} I)(v_{j+1}) = 0$, it follows that

$$\begin{aligned} 0 &= (L - \lambda_{j+1} I)(v_1 + \cdots + v_{j+1}) \\ &= (\lambda_1 - \lambda_{j+1})v_1 + \cdots + (\lambda_j - \lambda_{j+1})v_j \end{aligned}$$

- By the inductive assumption,

$$(\lambda_1 - \lambda_{j+1})v_1 = \cdots = (\lambda_j - \lambda_{j+1})v_j = 0$$

- Since $\lambda_i - \lambda_{j+1} \neq 0$ for each $1 \leq i \leq j$,

$$v_1 = \cdots = v_j = 0$$

- By (1), it follows that $v_{j+1} = 0$

Eigenspaces of Distinct Eigenvalues are Linearly Independent (Part 3)

- By induction,

$$v_1 + \cdots + v_k = 0 \implies v_1 = \cdots = v_k = 0$$

- This implies that $E_{\lambda_1}, \ldots, E_{\lambda_k}$ are linearly independent

Diagonalizability of a Linear Transformation (Part 1)

- Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $L : V \to V$
- **Theorem.** *L is diagonalizable if and only if*

$$\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_k}) = \dim V$$

- Let $n_0 = N_0 = 0$ and, for each $j \in \{1, \ldots, k\}$, let

$$n_j = \dim(E_{\lambda_j}) \text{ and } N_j = n_1 + \cdots + n_j$$

- In particular, for each $j \in \{1, \ldots, k\}$

$$N_j - N_{j-1} = n_j$$

  and $N_k = n_1 + \cdots + n_k = n$
- For each $j \in \{1, \ldots, k\}$, let

$$(v_{N_{j-1}+1}, \cdots, v_{N_j})$$

  be a basis of E_{λ_j}

Diagonalizability of a Linear Transformation (Part 2)

- Suppose

$$a^1 v_1 + \cdots + a^n v_n = 0,$$

- For each $j \in \{1, \ldots, k\}$, let

$$w_j = a^{N_{j-1}+1} v_{N_{j-1}} + \cdots + a^{N_j} v_{N_j} \in E_{\lambda_j}$$

- Since the eigenspaces are linearly independent and $w_1 + \cdots + w_k = 0$, it follows that

$$w_1 = \cdots = w_k = 0$$

- In particular, for each $j \in \{1, \ldots, k\}$,

$$0 = w_j = a^{N_{j-1}+1} v_{N_{j-1}} + \cdots + a^{N_j} v_{N_j},$$

which implies $a^{N_{j-1}+1} = \cdots = a^{N_j} = 0$

- Therefore, $(v_1, \ldots, v_n)$ is a basis of V
- This shows that L is diagonal with respect to this basis

Dot Product on $\mathbb{R}^n$

- Recall that the **dot product** of

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}, \; w = \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix} \in \mathbb{R}^n$$

  is defined to be

$$v \cdot w = v^1 w^1 + \cdots + v^n w^n = v^T w = w^T v$$

- The **norm** or **magnitude** of $v \in \mathbb{R}^n$ is defined to be

$$|v| = \|v\| = \sqrt{v \cdot v}$$

- If v and w are nonzero and the angle at 0 from v to w is θ, then

$$\cos\theta = \frac{v \cdot w}{|v||w|}$$

Properties of Dot Product

- The dot product is **bilinear** because for any $a, b \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$,

$$(au + bv) \cdot w = a(u \cdot w) + b(v \cdot w)$$
$$u \cdot (av + bw) = a(u \cdot v) + b(u \cdot w)$$

- It is **symmetric**, because for any $v, w \in \mathbb{R}^n$,

$$v \cdot w = w \cdot v$$

- It is **positive definite**, because for any $v \in \mathbb{R}^n$,

$$v \cdot v \geq 0$$

and

$$v \cdot v = 0 \iff v = 0$$

Inner Product on Real Vector Space

- Let V be an n-dimensional real vector space
- Consider a function

$$\alpha : V \times V \to \mathbb{R}$$

- It is **bilinear** if for any $a, b \in \mathbb{R}$ and $u, v, w \in V$,

$$\alpha(au + bv, w) = a\alpha(u, w) + b\alpha(v, w)$$
$$\alpha(u, av + bw) = a\alpha(u, v) + b\alpha(u, w)$$

  A bilinear function is also called a 2-tensor
- It is **symmetric** if for any $v, w \in V$,

$$\alpha(v, w) = \alpha(w, v)$$

- It is **positive definite** if for any $v \in V$,

$$\alpha(v, v) \geq 0$$

  and

$$\alpha(v, v) > 0 \iff v \neq 0$$

- Any positive definite symmetric 2-tensor on a **real** vector space V is called an **inner product**

Standard Hermitian Inner Product on $\mathbb{C}^n$

- Recall that if $z = x + iy \in \mathbb{C}$, then its **conjugate** is

$$\bar{z} = x - iy$$

If $z, w \in \mathbb{C}$, then $\overline{zw} = \bar{z}\bar{w}$ and **magnitude** is $|z| \geq 0$, where

$$|z|^2 = z\bar{z} = \bar{z}z = x^2 + y^2$$

- If $z, w \in \mathbb{C}$, then $\overline{zw} = \bar{z}\bar{w}$
- The standard Hermitian inner product on $\mathbb{C}^n$ of

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}, \quad w = \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}$$

is defined to be

$$(v, w) = \bar{w}^1 v^1 + \cdots + \bar{w}^n v^n = \bar{w}^T v = w^* v \in \mathbb{C},$$

where $w^* = \bar{w}^T$

Not a Real Inner Product

- **Not** bilinear, because if $c \in \mathbb{C}$,

$$(v, cw) = \bar{c}(v, w)$$

- **Not** symmetric, because

$$(w, v) = \overline{(v, w)}$$

- It is positive definite, because for any $v \in \mathbb{C}^n$,

$$(v, v) = v^1\bar{v}^1 + \cdots + v^n\bar{v}^n = |v^1|^2 + \cdots + |v^n|^2 \geq 0,$$

  and

$$(v, v) = 0 \iff v = 0$$

- The **norm** of $v \in \mathbb{C}^n$ is defined to be

$$|v| = \|v\| = \sqrt{(v, v)}$$

Properties of Hermitian Inner Product on $\mathbb{C}^n$

- It is a **linear** function of the first argument, because for any $a, b \in \mathbb{C}$, $u, v, w \in \mathbb{C}^n$,

$$(au + bv, w) = a(u, w) + b(v, w)$$

- It is a **conjugate linear** function of the second argument, which means that for any $a, b \in \mathbb{C}$, $u, v, w \in \mathbb{C}^n$,

$$(w, au + bv) = \bar{a}(w, u) + \bar{b}(w, v)$$

- It is **Hermitian**, which means

$$(v, w) = \overline{(w, v)}$$

- **No** geometric interpretation of the Hermitian inner product

Hermitian Inner Product of a Complex Vector Space

- An **inner product** over a complex vector space V is positive definite Hermitian 2-tensor

$$(\cdot, \cdot) : V \times V \to \mathbb{F}$$

- In other words, for any $a, b \in F$ and $u, v, w \in V$,

$$\begin{aligned}(au + bv, w) &= a(u, w) + b(v, w) \\ (w, v) &= \overline{(v, w)} \\ (v, v) &\geq 0 \\ (v, v) \neq 0 &\iff v \neq 0\end{aligned}$$

Examples of Inner Products on Real Vector Spaces

- Standard inner product on $\mathbb{R}^n$
- Space of polynomials of degree n or less with real coefficients with the inner product

$$(f, g) = \int_{t=0}^{t=1} f(t)g(t)\, dt$$

- Space of real matrices with n rows and m columns with the inner product

$$(A, B) = \operatorname{trace}(B^T A) = \sum_{1 \le k \le m} \sum_{1 \le j \le n} B^j_k A^j_k$$

Examples of Hermitian Inner Products on Complex Vector Spaces

- Standard Hermitian inner product on $\mathbb{C}^n$
- Space of polynomials of degree n or less with complex coefficients with the inner product

$$(f,g) = \int_{t=0}^{t=1} f(t)\overline{g(t)}\,dt$$

- Space of complex matrices with n rows and m columns with the inner product

$$(A,B) = \text{trace}(B^*A) = \sum_{1\le k\le m}\sum_{1\le j\le n} \bar{B}^j_k A^j_k,$$

Nondegeneracy Property

- Fact: If a vector $v \in V$ satisfies the following property:

$$\forall w \in V, \ (v, w) = 0,$$

then $v = 0$

- Corollary: If $v_1, v_2 \in V$ satisfy the property that

$$\forall w \in V, \ (v_1, w) = (v_2, w),$$

then $v_1 = v_2$

- Corollary: If '$L_1, L_2 : V \to W$ are linear maps such that

$$\forall v \in V, \ w \in W, \ (L_1(v), w) = (L_2(v), w),$$

then $L_1 = L_2$

Fundamental Inequalities

- **Cauchy-Schwarz inequality:** For any $v, w \in V$,

  $$|(v, w)| \leq |v||w|$$

  and

  $$|(v, w)| = |v||w|$$

  if and only if there exists $s \in \mathbb{F}$ such that

  $$v = sw \text{ or } w = sv$$

- **Triangle inequality:** For any $v, w \in V$,

  $$|v + w| \leq |v| + |w|$$

  and

  $$|v + w| = |v| + |w|$$

  if and only if there is a positive real scalar r such that $v = rw$ or $w = rv$

Proof of Triangle Inequality

- The triangle inequality follows easily from Cauchy-Schwarz inequality

$$
\begin{aligned}
|v+w|^2 &= (v+w, v+w) \\
&= |v|^2 + (v,w) + (w,v) + |w|^2 \\
&\leq |v|^2 + |(v,w)| + |(w,v)| + |w|^2 \\
&\leq |v|^2 + 2|v||w| + |w|^2 \\
&= (|v| + |w|)^2
\end{aligned}
$$

- If $|v+w| = |v| + |w|$, then

$$|(v,w)| = |(v,w)| = |v||w|,$$

- If $w \neq 0$, then there exists a scalar t such that $v = tw$
- Therefore,

$$|t+1|^2|w|^2 = |tw+w|^2 \quad = |tw|^2 + |w|^2 = (|t|^2+1)|w|^2,$$

which implies that $t = \bar{t}$, i.e., $t \in \mathbb{R}$

Polarization Identities

- On $\mathbb{R}^n$

$$(v, w) = \frac{1}{4}(|v + w|^2 - |v - w|^2)$$

- On $\mathbb{C}^n$

$$(v, w) = \frac{1}{4}(|v + w|^2 + i|v + iw|^2 - |v - w|^2 - i|v - iw|^2)$$

Norm Defined by Inner Product

- The norm of $v \in V$,
$$|v| = \sqrt{(v, v)}$$
satisfies the following properties for any $s \in \mathbb{F}$, $v, w \in V$
$$|sv| = |s||v| \qquad \text{(Homogeneity)}$$
$$|v| \geq 0 \qquad \text{(Nonnegativity)}$$
$$|v| = 0 \iff v = 0 \qquad \text{(Nondegeneracy)}$$
$$|v + w| \leq |v| + |w| \qquad \text{(Triangle inequality)}$$
- Homogeneity and the triangle inequality imply convexity: For any $0 \leq t \leq 1$ and $v, w \in V$,
$$|(1 - t)v + tw| \leq (1 - t)|v| + t|w|$$

Norm

- A norm on a vector space V over $\mathbb{F}$ is a function

$$g : V \to \mathbb{R},$$

that satisfies for any $s \in \mathbb{F}$ and $v, w \in V$,

$$
\begin{aligned}
|sv| &= |s||v| && \text{(Homogeneity)} \\
|v| &\geq 0 && \text{(Nonnegativity)} \\
|v| &= 0 \iff v = 0 && \text{(Nondegeneracy)} \\
|v + w| &\leq |v| + |w| && \text{(Triangle inequality)}
\end{aligned}
$$

Examples of Norms

- Given $1 \leq p < \infty$, the ℓ_p norm of $v \in \mathbb{F}^n$ is defined to be

$$|v|_p = (|v^1|^p + \cdots + |v^n|^p)^{1/p}$$

- The ℓ_∞ norm of $v \in \mathbb{F}^n$ is defined to be

$$|v|_\infty = \max(|v^1|, \ldots, |v^n|) = \lim_{p \to \infty} |v|_p$$

- The L_p norm of a continuous function $f : [0,1] \to \mathbb{C}$ is defined to be

$$\|f\|_p = \left(\int_{x=0}^{x=1} |f(x)|^p \, dx \right)^{1/p}$$

- The L_∞ norm of a continuous function $f : [0,1] \to \mathbb{C}$ is defined to be

$$\|f\|_\infty = \sup\{|f(x)| \ : \ 0 \leq x \leq 1\} = \lim_{p \to \infty} \|f\|_p$$

Parallelogram Identity

- A norm $|\cdot|$ on a vector space V satisfies the parallelogram identity

$$|v+w|^2 + |v-w|^2 = 2(|v|^2 + |w|^2),\ \forall v, w \in V$$

if and only if there is an inner product on V such that

$$|v|^2 = (v, v)$$

Orthogonality For Standard Dot Product on $\mathbb{R}^n$

- The following are synonyms: orthogonal, perpendicular, normal
- On $\mathbb{R}^n$,
  - Two vectors v_1, v_2 are called **orthogonal** if

$$v_1 \cdot v_2 = 0$$

  - A basis $(v_1, \ldots, v_n)$ is called **orthonormal** if for any $1 \leq i, j \leq n$,

$$v_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Orthogonality on an Inner Product Space

- Let V be an n-dimensional vector space over $\mathbb{F}$ with inner product $(\cdot, \cdot)$
- Two vectors v_1, v_2 are **orthogonal** if

$$(v_1, v_2) = 0$$

- Vectors $v_1, \ldots, v_k$ are **mutually orthogonal** if for every $1 \leq i < j \leq k$,

$$(v_i, v_j) \neq 0$$

- A set of nonzero muturally orthogonal vectors is called an **orthogonal set**

Linear Independence of Orthogonal Set

- An orthogonal set is linearly independent, because if
$$a^1 v_1 + \cdots + a^k v_k = 0,$$
then for any $j \in \{1, \ldots, k\}$,
$$0 = (v_j, a^1 v_1 + \cdots + a^k v_k) = a^j (v_j, v_j)$$
Since $v_j \neq 0$, $(v_j, v_j) \neq 0$ and therefore $a^j = 0$
- If
$$v = a^1 v_1 + \cdots + a^k v_k,$$
then for each $j \in \{1, \ldots, k\}$,
$$a^j = \frac{(v, v_j)}{|v_j|}$$
and
$$v = \frac{(v, v_1)}{|v_1|} + \cdots + \frac{(v, v_k)}{|v_k|}$$
- Any orthogonal set of n vectors is a basis

Orthonormal Set and Basis

- $\{v_1, \ldots, v_k\} \subset V$ is called an **orthonormal** set if for any $1 \leq i, j \leq k$,
$$(v_i, v_j) = \delta_{ij}$$
- If $\mathbb{F} = \mathbb{C}$, such a set is also called a **unitary** set
- An orthonormal set of n elements is called an **orthonormal** or **unitary** basis
- Any orthogonal set $\{v_1, \ldots, v_k\}$ can be turned into an orthonormal set,
$$\left\{ \frac{v_1}{|v_1|}, \ldots, \frac{v_k}{|v_k|} \right\}$$
- An orthormal or unitary basis is an orthonormal set with n elements,
$$E = (e_1, \ldots, e_n) \subset V$$
- If $v = a^1 e_1 + \cdots + a^n e_n$, then
$$a_j = (v, e_j)$$
- I.e.,
$$v = (v, e_1)e_1 + \cdots + (v, e_n)e_n$$

Example: Finite Fourier Decomposition (Part 1)

- For each $-N \leq k \leq N$, consider

$$v_k : [0, 2\pi] \to \mathbb{C}$$
$$\theta \mapsto e^{ik\theta}$$

- Let

$$V = \{a^{-N} v_N + \cdots + a^0 + \cdots + a^N v_N \ : \ (a^1, \ldots, a^N) \in \mathbb{C}^{2N+1}\}.$$

- V is a $(2N+1)$-dimensional complex vector space
- Consider the inner product

$$(f_1, f_2) = \int_{\theta=0}^{\theta=2\pi} f_1(\theta)\bar{f_2}(\theta)\, d\theta$$

Finite Fourier Decomposition (Part 2)

- If $j \neq k$, then

$$\begin{aligned}(v_j, v_k) &= \int_{\theta=0}^{\theta=2\pi} e^{i(j-k)\theta}\, d\theta \\ &= \left.\frac{e^{i(j-k)\theta}}{i(j-k)}\right|_{\theta=0}^{\theta=2\pi} \\ &= 0 \\ (v_k, v_k) &= \int_{\theta=0}^{\theta=2\pi} 1\, d\theta \\ &= 2\pi\end{aligned}$$

- Therefore, $(v_{-N}, \ldots, v_N)$ is an orthogonal basis, and $(u_{-N}, \ldots, u_N)$, where

$$u_k = \frac{v_k}{\sqrt{2\pi}}, \; -N \leq k \leq N,$$

is an orthonormal basis

Finite Fourier Decomposition (Part 3)

- Given any $f : C^0([0, 2\pi])$, let

$$f_N(\theta) = a^{-N} u_{-N} + \cdots + a^N u_N,$$

where

$$a^k = (f, u_k) = \frac{1}{\sqrt{2\pi}} \int_{\theta=0}^{\theta_2\pi} f(\theta) e^{-ik\theta}\, d\theta$$

- When is f_N is a good approximation to f?
- When is

$$f = \sum_{k=-\infty}^{k=\infty} a^k u_k?$$