MA-GY 7043: Linear Algebra II

Fundamental Subspaces of Adjoint Map
Isometries
Unitary Transformations and Matrices

Deane Yang

Courant Institute of Mathematical Sciences
New York University

March 9 2026

Outline I

Unitary Transformations and Matrices

Unitary Transformations and Matrices

Fundamental Subspaces of Adjoint Map

- Let $L : V \to W$ be a map between inner produt spaces
- Then

$$\ker(L^*) = (\text{image}(L))^\perp \tag{1}$$
$$\ker(L) = (\text{image}(L^*))^\perp \tag{2}$$
$$\text{image}(L) = (\ker(L^*))^\perp \tag{3}$$
$$\text{image}(L^*) = (\ker(L))^\perp \tag{4}$$

- That
  - For any subspace S, $(S^\perp)^\perp = S$
  - For any linear map A, $(A^*)^* = A$

  imply that (2),(3),(4) follow directly from (1)

Proof that $\ker(L^*) = (\text{image}(L))^\perp$

$$
\begin{aligned}
w \in \ker(L^*) &\iff L^*(w) = 0 \\
&\iff \forall v \in V,\ (v, L^*(w)) = 0 \\
&\iff \forall v \in V,\ (L(v), w) = 0 \\
&\iff w \in (\text{image}(L))^\perp
\end{aligned}
$$

Geometric Description of a Linear Map and its Adjoint

- Recall that if E is a subspace of V, then

$$V = E \oplus E^{\perp}$$

- Therefore,

$$V = (\ker(L)) \oplus (\ker(L))^{\perp}$$

- It is easy to show that the restriction of L to $(\ker(L))^{\perp}$,

$$L : (\ker(L))^{\perp} \to \text{image}(L)$$

  is bijective
- Equivalently, by (4),

$$L : \text{image}(L^*) \to \text{image}(L)$$

  is bijective
- Therefore,

$$\text{rank}(L) = \dim(\text{image}(L)) = \dim(\text{image}(L^*)) = \text{rank}(L^*)$$

Isometries

- A map (not assumed to be linear) $L : V \to W$, where V and W are normed vector spaces, is an **isometry** if for any $v_1, v_2 \in V$,

$$|L(v_1) - L(v_2)| = |v_2 - v_1|$$

- **Theorem:** If V and W are inner product spaces and $L : V \to W$ is an isometry, then L is linear and satisfies for any $v_1, v_2 \in V$,

$$(L(v_1), L(v_2)) = (v_1, v_2)$$

- **Lemma:** $L : V \to W$ is an isometry if and only if $L^* \circ L = I_V$, i.e., L^* is a left inverse of L
- In particular, if $L(v) = 0$, then

$$v = L^*(L(v)) = 0$$

  and therefore, $\ker(L) = \{0\}$
- It follows that if $L : V \to W$ is an isometry, then

$$\dim(V) \leq \dim(W)$$

Basic Properties of Isometries

- If $L : V \to W$ is an isometry and $(v_1, \ldots, v_n)$ is an unitary basis of V, then $(L(v_1), \ldots, L(v_n))$ is an unitary set in W
- If $L_1 : V \to W$ and $L_2 : W \to X$ are unitary, then so is $L_2 \circ L_1 : V \to X$

Unitary Transformation

- If $W = V$, then an isometry $L : V \to V$ is called a **unitary transformation**
- If V is an inner product space, a linear transformation $L : V \to V$ is **unitary**, if for any $v, w \in V$, if any of the following equivalent statements hold:

$$(L(v), L(w)) = (v, w)$$
$$(L^* L(v), w) = (v, w)$$
$$L^* \circ L = I$$
$$L \text{ is invertible and } L^{-1} = L^*$$

Unitary Matrices

- Let $L : V \to V$ be a unitary map
- If $(u_1, \ldots, u_n)$ is a unitary basis of V and $L(u_k) = M^j_k u_j$, then

$$\begin{aligned}\delta_{jk} &= (u_j, u_k) \\ &= (L(u_j), L(u_k)) \\ &= (u_j, (L^* \circ L)(u_k)) \\ &= (u_j, (M^* M)^i_k u_i) \\ &= (M^* M)^j_k\end{aligned}$$

- $$M^* M = I$$
- A matrix M is **unitary** if $M^* M = M M^* = I$

Examples of Unitary Matrices

- An n-by-n matrix is unitary if and only if its columns form a unitary basis of $\mathbb{F}^n$
- A real 2-by-2 matrix is a unitary matrix with positive determinant if and only if it is of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- For any $\theta^1, \theta^2 \in \mathbb{R}$

$$\begin{bmatrix} e^{i\theta^1} & 0 \\ 0 & e^{i\theta^2} \end{bmatrix}$$

More Properties of Unitary Matrices

- Let U be a unitary matrix
- $\det(U^*) = \overline{\det(U)}$
  - Because $\det(A^T) = \det(A)$ and $\det(\overline{A}) = \overline{\det(A)}$
- If λ is an eigenvalue of U, then $|\lambda| = 1$
  - Because if λ is an eigenvalue of U with eigenvector v, then

$$|v| = |Uv| = |\lambda v| = |\lambda||v|,$$

which implies $|\lambda| = 1$

Properties of unitary maps and matrices

- If L_1, L_2 are unitary maps, then so is $L_1 \circ L_2$
  - If M_1, M_2 are unitary matrices, then so is $M_1 M_2$
- If L is unitary, then L is invertible and $L^{-1} = L^*$ is unitary
  - If M is unitary, then M is invertible and $M^{-1} = M^*$ is unitary
- The identity map is unitary
  - The identity matrix is unitary

Unitary Group

- Define the unitary group $U(V)$ of a Hermitian vector space V to be the set of all unitary transformations
- Denote
$$U(n) = U(\mathbb{C}^n)$$
using the standard Hermitian inner product on $\mathbb{C}^n$
- Both satisfy the properties of an abstract group G
  - Any ordered pair $(g_1, g_2) \in G \times G$ uniquely determine a third, denoted $g_1 g_2 \in G$
  - (Associativity) $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
  - (Identity element) There exists an element $e \in G$ such that $ge = eg = g$ for any $g \in G$
  - (Inverse of an element) For each $g \in G$, there exists an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1} g = e$
- $U(n)$ is an example of a matrix group
- Both $U(V)$ and $U(n)$ are examples of Lie groups