Unitary Diagonalization

Schur Decomposition

# MA-GY 7043: Linear Algebra II

## Unitary Equivalence of Matrices
## Unitary Diagonalization
## Schur Decomposition

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# Outline I

Unitary Diagonalization

Schur Decomposition

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# Unitarily Equivalent Matrices

Unitary Diagonalization

Schur Decomposition

- $M_1, M_2 \in \mathrm{gl}(n, \mathbb{F})$ are **unitarily equivalent** if there exists a unitary matrix $U$ such that

$$M_2 = UM_1U^*$$

- Since $U^* = U^{-1}$, unitarily equivalent implies similar

# Unitary Diagonalization of Linear Transformations and Matrices

Unitary Diagonalization

Schur Decomposition

- Recall that a linear transformation $L : V \to V$ is diagonalizable if there exists a basis of eigenvectors
- Equivalently, a matrix $M$ is diagonalizable if and only if there is a basis of eigenvectors
- **Definition:** A linear transformation $L : V \to V$ is **unitarily diagonalizable** if there exists a unitary basis of eigenvectors
- **Definition:** A square matrix $M$ is **unitarily diagonalizable** if there exists a unitary basis of eigenvectors
- Equivalently, $M$ is unitarily diagonalizable if and only if there exists a unitary matrix $U$ and diagonal matrix $D$ such that

$$M = UDU^*$$

# Schur Decomposition of a Real Linear Map



- Let $V$ be a finite dimensional real inner product space
- **Theorem:** Given any linear map $L : V \to V$ with only real eigenvalues, there exists an **orthonormal** basis $E = (e_1, \ldots, e_n)$ of $V$ such that for each $1 \leq k \leq n$, $L(e_k)$ is a linear combination of $e_1, \ldots, e_k$,
$$L(e_k) = e_k M_k^k + \cdots + e_n M_k^n$$
- In other words, there exists an orthonormal basis $E$ such that
$$L(E) = EM,$$
where $M$ is a lower triangular matrix.
- **Corollary:** Given any real matrix $M$ with only real eigenvalues, there is an orthogonal matrix $O$ such that the matrix $O^t MO$ is lower triangular

# Schur Decomposition of a Complex Linear Map



- **Theorem:**
  - Let $V$ be an $n$-dimensional Hermitian vector space over $\mathbb{F}$
  - Let $L : V \to V$ be a linear map with $n$ eigenvalues in $\mathbb{F}$, counting multiplicity

  Then there exists a **unitary** basis $U = (u_1, \ldots, u_n)$ of $V$ such that for each $1 \leq k \leq n$, $L(u_k)$ is a linear combination of $u_1, \ldots, u_k$,

$$L(u_k) = u_k M_k^k + \cdots + u_n M_k^n$$

- In other words, there exists a unitary basis $U$ such that

$$L(U) = UM,$$

  where $M$ is a lower triangular matrix
- **Corollary:** Given any complex matrix $M$, there is a unitary matrix $U$ such that the matrix $U^* MU$ is triangular

# Proof (Part 1)



- Proof by induction
- Theorem holds when $\dim V = 1$
- Suppose theorem holds when $\dim V = n - 1$
- Consider a linear map $L : V \to V$, where $\dim V = n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$
- Let $u_n$ be a unit eigenvector for the eigenvalue $\lambda_n$, i.e.,

$$\|u_n\| = 1 \text{ and } L(u_n) = \lambda_n u_n$$

- Let

$$u_n^{\perp} = \{v \in V \ : \ (v, u_n) = 0\}$$

- Recall that the orthogonal projection maps onto $[u_n]$ and $u_n^{\perp}$ respectively are

$$\begin{aligned} \pi : V &\to [u_n] \\ v &\mapsto (v, u_n)u_n \\ \pi^{\perp} : V &\to u_n^{\perp} \\ v &\mapsto v - u_n(v, u_n) \end{aligned}$$

# Proof (Part 2)



- If $(v_1, \ldots, v_{n-1})$ is a basis of $u_n^{\perp}$, then $(v_1, \ldots, v_{n-1}, u_n)$ is a basis of $V$
- Let $M$ be the matrix such that for $1 \leq k \leq n-1$,

$$L(v_k) = v_1 M_k^1 + \cdots + v_{n-1} M_k^{n-1} + u_n M_k^n$$

and

$$L(u_n) = v_1 M_n^1 + \cdots + v_{n-1} M_n^{n-1} + u_n M_n^n$$

# Proof (Part 3)



- Since $L(u_n) = \lambda_n u_n$,

$$M_n^1 = \cdots = M_n^{n-1} = 0 \text{ and } M_n^n = \lambda_n$$

and therefore $M$ is of the form

$$M = \left[\begin{array}{ccc|c} M_1^1 & \cdots & M_{n-1}^1 & M_n^1 \\ \vdots & & \vdots & \vdots \\ M_1^{n-1} & \cdots & M_{n-1}^{n-1} & M_n^{n-1} \\ \hline 0 & \cdots & 0 & M_n^n \end{array}\right]$$

- Let

$$\widehat{M} = \begin{bmatrix} M_1^1 & \cdots & M_{n-1}^1 \\ \vdots & & \vdots \\ M_1^{n-1} & \cdots & M_{n-1}^{n-1} \end{bmatrix}$$

- It follows that

$$p_L(x) = \det(L - xI) = \det(M - xI) = \det(\widehat{M} - xI)(M_n^n - x)$$

- In particular, the eigenvalues of $\widehat{M}$ are eigenvalues of $M$

# Proof (Part 4)



- Let $L^{\perp} : u_n^{\perp} \to u_n^{\perp}$ be the linear transformation given by

$$L^{\perp}(v) = \pi^{\perp}\left(L|_{u_n^{\perp}}(v)\right)$$

- By assumption, $L^{\perp}$ has a Schur decomposition, i.e., a unitary basis $u_1, \ldots, u_{n-1}$ and a lower triangular matrix $\widehat{M}$ such that for each $1 \leq k \leq n-1$,

$$L^{\perp}(u_k) = u_k \widehat{M}_k^k + \cdots + u_{n-1}\widehat{M}_k^{n-1}$$

and therefore

$$L(u_k) = u_k \widehat{M}_k^k + \cdots + u_{n-1}\widehat{M}_k^{n-1} + u_n M_k^n$$

- Also,

$$L(u_n) = \lambda_n u_n$$

- Therefore,

$$L(u_k) = M_k^k u_k + \cdots + M_k u^{n-1} u_{n-1} + M_k^n u_n,\ 1 \leq k \leq n,$$

where $M_n^n = \lambda_n$

- This proves the theorem