Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

MA-GY 7043: Linear Algebra II

Functions of Linear Transformations and Matrices
Self-Adjoint Transformations and Matrices

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Outline I

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Integer Powers of Linear Transformations

Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

- Let V be a vector space (without inner product) and $L : V \to V$ be a linear transformation
- Denote $L^0 = I$
- For each positive integer k, denote

$$L^k = L \circ \cdots \circ L$$

- L is invertible if and only if L^k is
- For any $j, k \in \mathbb{Z}$,

$$L^j \circ L^k = L^{j+k}$$

Powers of Commuting Linear Transformations

Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

- Two linear transformations ${}_1L, L_2$ of V **commute** if

$$L_1 \circ L_2 = L_2 \circ L_1$$

- If so, for $k \in \mathbb{Z}$,

$$(L_1 \circ L_2)^k = L_1^k \circ L_2^k$$

- This is false if L_1 and L_2 do not commute

Exponential of Linear Transformations and Matrices

Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

- Recall that
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
- We can therefore try to define the exponential of a linear transformation L to be
$$e^L = \sum_{k=0}^{\infty} \frac{L^k}{k!}$$
- Using analysis, it can be proved that for each $v \in V$ that the sum
$$\sum_{k=0}^{N} \frac{L^k(v)}{k!}$$
converges as $N \to \infty$
- The exponential of a matrix is defined to be
$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

Exponential of Commuting Linear Transformations

Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

- If linear transformations L_1, L_2 commute, then

$$e^{L_1+L_2} = e^{L_1} e^{L_2}$$

- This is false if L_1 and L_2 do not commute

Powers of a Diagonalizable Linear Transformation

Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

- Let L be a diagonalizable linear transformation with a basis of eigenvectors $(e_1, \ldots, e_n)$ and respective eigenvalues $(\lambda_1, \ldots, \lambda_n)$
- It follows that for any nonnegative integer p,

$$L^p(e_j) = \lambda_j^p e_j$$

  and therefore L^p has the same basis of eigenvectors with eigenvalues $(\lambda_1^p, \ldots, \lambda_n^p)$
- If L is invertible, then $(e_1, \ldots, e_n)$ is also a basis of eigenvectors for L^{-1} with eigenvalues $(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$
- It follows that if L is invertible, then for any integer p, L^p is invertible and

$$(L^p)^{-1} = (L^{-1})^p$$

Exponential of Diagonalizable Transformation

- If $L : V \to V$ has a basis of eigenvalues $(e_1, \ldots, e_n)$ with corresponding eigenvalues $(\lambda_1, \ldots, \lambda_n)$, then for each $k \in \{1, \ldots, n\}$,

$$\begin{aligned} e^L(e_k) &= \left(\sum_{j=0}^{\infty} \frac{L^j}{j!} \right) (e_k) \\ &= \sum_{j=0}^{\infty} \frac{L^j(e_k)}{j!} \\ &= \sum_{j=0}^{\infty} \frac{\lambda_k^j e_k}{j!} \\ &= \sum_{j=0}^{\infty} \frac{\lambda_k^j}{j!} e_k \\ &= e^{\lambda_k} e_k \end{aligned}$$

Roots of Positive Diagonalizable Transformations

- Let $L : V \to V$ be diagonalizable with nonnegative real eigenvalues
- Let $(e_1, \ldots, e_n)$ be a basis of eigenvectors with respective eigenvalues $(\lambda_1, \ldots, \lambda_n)$
- For each nonzero integer p, define $R_p : V \to V$ by

$$R_p(e_k) = \lambda_k^{1/p} e_k$$

- It follows that $R_p^p = L$
- Therefore, we denote R_p by $L^{1/p}$
- If L is invertible, then $L^{1/p}$ is invertible and

$$(L^{1/p})^{-1} = (L^{-1})^{1/p}$$

- If L_1, L_2 are commuting linear transformations, then for any rational q,

$$(L_2 \circ L_1)^q = L_2^q \circ L_1^q = L_1^q \circ L_2^q$$

Logarithm of Positive Diagonalizable Transformations

Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

- Let $L : V \to V$ have a basis of eigenvectors $(e_1, \ldots, e_n)$ with positive eigenvalues $(\lambda_1, \ldots, \lambda_n)$
- Define the **logarithm** of L to be the linear transformation $\log(L)$, where
$$\log(L)(e_k) = e_k \log(\lambda_k)$$
- The following holds:
$$e^{\log(L)} = \log(e^L) = L$$
- For any $p \in \mathbb{R}$ and $x \in (0, \infty)$, since $x^p = e^{p \log x}$, we can also define
$$L^p = e^{p \log(L)}$$

Function of a Diagonalizable Transformation

- Consider a function $f : \Omega \to \mathbb{F}$, where $\Omega \subset \mathbb{F}$
- Let $L : V \to V$ have a basis of eigenvectors $(e_1, \ldots, e_n)$ with corresponding eigenvalues $(\lambda_1, \ldots, \lambda_n)$
- If $\lambda_1, \ldots, \lambda_n \in \emptyset$, then we can define the linear transformation $f(L)$ by setting, for each $k \in \{1, \ldots, n\}$,

$$f(L)(e_k) = f(\lambda_k)e_k$$

Functions of Diagonal Matrices

- If D_1, D_2 are diagonal matrices, then so are $D_1 + D_2, D_1 D_2$
- If D is diagonal and a function $f : \Omega \to \mathbb{F}$, where $\Omega \subset \mathbb{F}$, then if each entry on the diagonal of D lies in Ω, we can define $f(D)$ by setting

$$f(D)e_k = f(D_k^k)e_k,$$

where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{F}^n$

Functions of Diagonalizable Matrices

Functions of Linear Transformations

Self-Adjoint Transformations and Matrices

- Let M be a diagonalizable matrix, i.e., there exists an invertible matrix S and diagonal matrix D such that

$$M = SDS^{-1}$$

- Recall that the columns of S comprise a basis of eigenvectors
- If $f : \Omega \to \mathbb{F}$ is a function, then, if the eigenvalues of M lie in Ω, we can define

$$f(M) = Sf(D)S^{-1}$$

Self-Adjoint Transformations and Symmetric Matrices

- Let V be an inner product space
- Recall that the **adjoint** of a linear transformation $L : V \to V$ is the map $L^* : V \to V$ such that for any $v_1, v_2 \in V$,

$$(L(v_1), v_2) = (v_1, L^*(v_2))$$

- A linear transformation $L : V \to V$ is **self-adjoint** if

$$L^* = L$$

- The adjoint of a square matrix M is

$$M^* = \overline{M}^T$$

- A square matrix is **self-adjoint** or **Hermitian** if

$$M^* = M$$

- A square real matrix is **symmetric** if

$$M^T = M$$

Eigenvalues of a Self-Adjoint Map are Real

- Let $L : V \to V$ be a hermitian linear map with basis $(e_1, \cdots, e_n)$
- If v is an eigenvector of L with eigenvalue λ, then

$$\begin{aligned}
\lambda \|v\|^2 &= (L(v), v) \\
&= (v, L(v)) \\
&= \overline{(L(v), v)} \\
&= \overline{\lambda \|v\|^2} \\
&= \overline{\lambda} \|v\|^2
\end{aligned}$$

Eigenspaces of a Self-Adjoint Map are Orthogonal

- Suppose λ, μ are two different eigenvalues of a self-adjoint operator $L : V \to V$ with eigenvectors v, w respectively
- It follows that

$$\begin{aligned} 0 &= (L(v), w) - (v, L(w)) \\ &= (\lambda v, w) - (v, \mu w) \\ &= (\lambda - \mu)(v, w) \text{ since } \mu \in \mathbb{R} \end{aligned}$$

- Since $\lambda - \mu \neq 0$, it follows that $(v, w) = 0$

Self-Adjoint Map Has Unitary Basis of Eigenvectors

- **Theorem.** Given a self-adjoint map $L : V \to V$, there exists a unitary basis of eigenvectors
- **Corollary.** Given a Hermitian matrix M, there exists a unitary matrix $U \in U(n)$ and real diagonal matrix D such that

$$M = UDU^*,$$

Proof of Theorem

- Given a linear map $L : V \to V$, by the Schur decomposition, there exists a unitary basis $(u_1, \ldots, u_n)$ such that

$$L(u_k) = u_k M_k^k + \cdots u_n M_k^n, \text{ for each } 1 \leq k \leq n$$

- Equivalently, for any $1 \leq k \leq n$ and $1 \leq j < k$,

$$(L(u_k), u_j) = 0$$

- If L is self-adjoint, then for any $1 \leq k \leq n$ and $1 \leq j < k$,

$$0 = (L(u_k), u_j) = (u_k, L^*(u_j)) = (u_k, L(u_j)) = \overline{(L(u_j), u_k)},$$

  which implies $(L(u_j), u_k) = 0$
- Therefore, for any $1 \leq k \leq n$ and $1 \leq j < k$,

$$(L(u_j), u_k) = 0$$

- It follows that for each $1 \leq k \leq n$,

$$L(u_k) = M_k^k u_k$$

Direct Proof of Corollary

- Given a square matrix M, by the Schur decomposition, there exists a unitary matrix U such that

$$M = UTU^*,$$

where T is upper triangular
- If M is self-adjoint, then

$$UTU^* = M = M^* = (U^*)^* T^* U^* = UT^* U^*$$

- Therefore,

$$T = U^* UTU^* U = U^*(UTU^*)U = U^*(UT^* U^*)U = T^*,$$

which implies T is self-adjoint
- Since T is upper triangular,

$$T_k^j = 0, \text{ if } j < k \leq n$$

- Since $T^* = T$,

$$T_j^k = (T^*)_j^k = \bar{T}_k^j = 0, \text{ if } j < k \leq n$$

- It follows that T is diagonal

Hermitian Inner Product With Respect To Basis

- Let V be a complex vector space and let $(f_1, \ldots, f_n)$ be a basis of V
- Any inner product on V is uniquely determined by the matrix A, where

$$A_{ij} = (f_i, f_j)$$

because if $v = f_j a^j$ and $w = f_k b^k$, then

$$(v, w) = (f_j a^j, f_k b^k) = (f_j, f_k) a^j \bar{b}^k = A_{jk} a^j \bar{b}^k$$

- The matrix A satisfies the following properties
  - Hermitian:

$$A_{ij} = (f_i, f_j) = \overline{(f_j, f_i)} = \bar{A}_{ji}$$

(In particular, since $A_{ii} = \bar{A}_{ii}$, it follows that $A_{ii} \in \mathbb{R}$)

  - Positive definite: If $v = f_k a^k$, then

$$0 \leq (v, v) = (f_j a^j, f_k a^k) = (f_j, f_k) a^j \bar{a}^k = A_{jk} a^j \bar{a}^k$$

and equality holds if and only $v = 0$, i.e., $a^1 = \cdots = a^n = 0$

Inner product $\iff$ Positive Definite Hermitian Matrix

- Conversely, if A is a positive definite hermitian matrix and $(f_1, \ldots, f_n)$ is a basis of a vector space V (not necessarily an inner product space), then the bilinear tensor $B : V \times V \to \mathbb{F}$ given by
$$B(f_j a^j, f_k b^k) = A_{jk} a^j b^k$$
is positive definite and hermitian
- It follows that B is an inner product if and only if A is a positive definite Hermitian matrix

Positive Definite Self-Adjoint Maps

- Let V be an inner product space
- A self-adjoint map $L : V \to V$ is **positive definite** if for any $v \neq 0$,
$$(L(v), v) > 0$$
- If L is a positive definite self-adjoint map, we write $L > 0$
- $L > 0$ if and only if the eigenvalues of L are all positive
- We write $L \geq 0$ if the eigenvalues of L are all nonnegative
- The function $Q : V \times V \to \mathbb{F}$ given by
$$Q(v, w) = (L(v), v)$$
is an inner product
- Let $(f_1, \ldots, f_n)$ be a basis of V
- If $A_{jk} = (L(f_j), f_k)$, $v = f_j a^j$, $w = f_k b^k$, then
$$Q(v, w) = (L(f_j), f_k) a^j \bar{b}^k = A_{jk} a^j \bar{b}^k$$
- It follows that Q is an inner product if and only if the matrix A is a positive definite hermitian matrix