MA-GY 7043: Linear Algebra II

Singular Value Decomposition

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Outline I

Singular Values of a Linear Map

- Let X and Y be inner product spaces (not necessarily of the same dimension)
- Let $L : X \to Y$ be *any* linear map
- The map $L^*L : X \to X$ is self-adjoint, because for any $x_1, x_2 \in X$,

$$(L^*(L(x_1)), x_2)_X = (L(x_1), L(x_2))_Y = (x_1, L^*(L(x_2)))_X$$

- $L^*L \geq 0$, because for any $x \in X$,

$$(L^*L(x), x) = (L(x), L(x)) \geq 0$$

- We can denote $|L| = \sqrt{L^*L}$
- The eigenvalues of $|L|$ are called the **singular values** of L
- Singular values are always real and nonnegative
- Since $\ker L^*L = \ker L$, if $k = \dim \ker L$, then exactly k singular values are zero

Normal Form of a Linear Map (Part 1)

- Let X and Y be vector spaces such that $\dim X = m$ and $\dim Y = n$
- Let $L : X \to Y$ be a linear map with rank r
- If $\dim \ker L = k$, then, by Rank Theorem, $r + k = m$
- Let $(u_{r+1}, \ldots, u_m)$ be a unitary basis of $\ker L$
- Since
$$|L| = \sqrt{L^* L}$$
is self-adjoint, this can be extended to a unitary basis of eigenvectors of $|L|$
- Therefore,
$$|L|(u_j) = s_j u_j, 1 \leq j \leq r,$$
where $s_1, \ldots, s_r$ are the singular values of L
- Observe that
$$s_1, \ldots, s_r > 0$$

Normal Form of a Linear Map (Part 2)

- Let $\tilde{v}_j = L(u_j)$, $1 \leq j \leq r$
- The set $\{\tilde{v}_1, \ldots, \tilde{v}_r\}$ is linearly independent because if

$$a^1 \tilde{v}_1 + \cdots + a^k \tilde{v}_r = 0,$$

then

$$L(a^1 u_1 + \cdots + a^r u_r) = a^1 \tilde{v}_1 + \cdots + a^r \tilde{v}_r = 0$$

which implies that $a^1 u_1 + \cdots + a^r u_r \in \ker L$ and therefore

$$a^1 u_1 + \cdots + a^r u_r = b^{r+1} u_{r+1} + \cdots + b^m u_m,$$

which implies that $a^1 = \cdots = a^r = b^{r+1} = \cdots = b^m = 0$

- Moreover, if $1 \leq i, j \leq r$, then $s_j > 0$ and therefore

$$(\tilde{v}_i, \tilde{v}_j) = (L(u_i), L(u_j)) = (u_i, (L^* L)(u_j)) = s_j^2 (u_i, u_j) = s_j^2 \delta_{ij}$$

Normal Form for a Linear Map (Part 3)

- It follows that if

$$v_j = s_j^{-1}\tilde{v}_j = s_j^{-1}L(u_j),\ j \in \{1, \ldots, k\},$$

  then $(v_1, \ldots, v_r)$ is a unitary basis of $L(X) \subset Y$ and therefore a unitary set in Y
- This can be extend to a unitary basis $(v_1, \ldots, v_n)$ of Y
- Therefore, there is a unitary basis $(u_1, \cdots, u_m)$ of X and a unitary basis $(v_1, \ldots, v_n)$ of Y such that

$$L(u_j) = \begin{cases} s_j v_j & \text{if } 1 \leq j \leq \dim \ker L \\ 0 & \text{if } j > \dim \ker L \end{cases}$$

Normal Form for a Linear Map (Part 4)

- $$\left[\begin{array}{ccc|ccc} L(u_1) & \cdots & L(u_r) & L(u_{r+1}) & \cdots & L(u_m) \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc} v_1 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{array}\right] \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right],$$
where D is the r-by-r diagonal matrix such that
$$D^i_j = \begin{cases} s_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- $$L(a^1 u_1 + \cdots + a^m u_m)$$
$$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right] \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}$$

Singular Value Decomposition of a Complex Matrix

- If M is an n-by-m complex matrix with rank r and whose positive singular values are $s_1, \ldots, s_r$, then there are unitary matrices $P \in U(m)$ and $Q \in U(n)$ such that

$$M = QSP,$$

where

$$S = \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right]$$

and D is the r-by-r diagonal matrix such that

$$D^i_j = \begin{cases} s_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Polar Decomposition of Linear Map

- Let X and Y be inner product spaces such that $\dim(X) = \dim(Y)$
- Consider a linear map
$$L : X \to Y$$
- Then there exists a unitary map $U : X \to Y$ such that
$$L = U|L|$$
- Proof: By the singular value decomposition of L,
$$L = W\Sigma V^* = (WV^*)V\Sigma V^* = U|L|$$