THEOREMA EGREGIUM OF GAUSS FOR SUBMANIFOLDS

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In this note, we describe a simple way to define the second fundamental form of an m-dimensional submanifold M in Euclidean space $\mathbb{R}^n$ and use it to prove Gauss's Theorema Egregium, as well as its analogue in higher dimensions.

The basic idea is to parameterize M in a neighborhood of $p \in M$ as the graph of a map

$$F : O \to T_p^\perp M,$$

where $O \subset T_pM$ is an open neighborhood of 0 and $T_p^\perp M$ is the orthogonal . The map F has a critical point at 0, and the second fundamental form at p is defined to be Hessian of F at p. It is a linear map

$$H_p : S^2T_pM \to T_p^\perp M \subset \mathbb{R}^n.$$

The statement of the Theorem Egregium then falls out of a straightforward calculation that seeks to identify a tensor invariant, defined in terms of the second fundamental form, that is independent of the isometric embedding. This tensor invariant is therefore an intrinsic geometric invariant of the hypersurface. Another straightforward tensor calculation leads to an intrinsic definition of Gauss curvature for a surface in $\mathbb{R}^3$ and, more generally, the Riemann curvature tensor.

1. The second fundamental form

Given $p \in M$, let $T_pM \subset \mathbb{R}^n$ be the tangent space of M and $T_p^\perp M \subset \mathbb{R}^n$ be the orthogonal complement. By the implicit function theorem, it follows that there is an open neighborhood O of $0 \in T_pM$ and a smooth map

$$F : O \to T_p^\perp M \subset \mathbb{R}^n$$

such that in a neighborhood O of p, M is the graph of F. In particular, $F(0) = 0$,

$$M \cap O \times T_p^\perp M = \{x + F(x) : \ x \in O\},$$

and $0 \in O$ is a critical point of F. This implies that for any $v \in T_pM \in \mathbb{R}^n$,

$$H_p(v, v) = \left.\frac{d^2}{dt^2}\right|_{t=0} F(tv).$$

is a well-defined normal-vector-valued symmetric 2-tensor at p. The *second fundamental form* of M at p is defined to be H_p. Since F is uniquely determined by M, p, and the Euclidean geometric structure of $\mathbb{R}^n$, so is H_p. Moreover, the definition of H_p is independent of coordinates. It is therefore a geometric invariant of submanifold M. Since it depends on the embedding of M, it is an extrinsic invariant.

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2. Isometric submanifolds

Let T denote $e_n^\perp$.

Two submanifolds M and $\widehat{M}$ are *isometric* if there exists a smooth diffeomorphism

$$\Phi : M \to \widehat{M}$$

such that the length of any smooth curve $C \subset M$ is equal to the length of the curve $\Phi(C) \subset \widehat{M}$.

Given $p \in M$, let $\hat{p} = \Phi(p) \in \widehat{M}$. Applying rigid motions to both M and $\widehat{M}$, we can assume that that $\hat{p} = p = 0$ and the respective tangent hyperplanes are

$$T_0\widehat{M} = T_0 M = T,$$

where

$$T = \{v \in \mathbb{R}^n : \ v^{k+1} = \cdots = v^n = 0\}.$$

Let $T^\perp$ denote the orthogonal complement of T. By the implicit function theorem, there exists an open neighborhood $O \subset T$ of 0 and smooth maps

$$F : O \to T^\perp, \ \widehat{F} : O \to T^\perp$$

such that

$$M \cap (O \times T^\perp) = \{(x, F(x) : \ x \in O\}$$
$$\widehat{M} \cap (O \times T^\perp) = \{(x, \widehat{F}(x) : \ x \in O\}.$$

Moreover, there is a diffeomorphism $\phi : O \to O$ such that, for each $x \in O$,

$$\Phi(x, F(x)) = (\phi(x), \widehat{F}(\phi(x))).$$

The diffeomorphism $\Phi : M \to \widehat{M}$ is an isometry if and only if for any $1 \leq i, j \leq m$,

$$\partial_i \hat{y} \cdot \partial_j \hat{y} = \partial_i y \cdot \partial_j y,$$

where, for each $x \in O$,

$$y(x) = (x, F(x))$$
$$\hat{y}(x) = (\phi(x), \hat{F}(\phi(x)).$$

Equivalently,

$$\partial_i \phi \cdot \partial_j \phi + \partial_i \phi^p \partial_p \hat{F} \cdot \partial_q \hat{F} \partial_j \phi^q = \delta_{ij} + \partial_i F \cdot \partial_j F. \tag{1}$$

Note that

$$\partial_i F(0) = \partial_i \hat{F}(0) = 0$$
$$\partial_i \phi^j(0) = \delta_i^j.$$

Therefore, if we differentiate (1) and evaluate at $x = 0$, we get

$$\partial^2_{ik} \phi^j + \partial^2_{jk} \phi^i = 0.$$

This implies that, at $x = 0$,

$$\partial^2_{ik} \phi^j = -\partial^2_{jk} \phi^i = \partial^2_{ji} \phi^k = -\partial^2_{ik} \phi^j, \tag{2}$$

and therefore

$$\partial^2_{jk} \phi^i(0) = 0, \text{ for all } 1 \leq i, j, k \leq m.$$

If we differentiate (1) again, evaluate at $x = 0$, and cycle through the indices i, j, k, l, we get

$$\partial^3_{ikl}\phi^j + \partial^3_{jkl}\phi^i + \partial^2_{ik}\hat{F}\cdot\partial^2_{jl}\hat{F} + \partial^2_{il}\hat{F}\cdot\partial^2_{jk}\hat{F} = \partial^2_{ik}F\cdot\partial^2_{jl}F + \partial^2_{il}F\cdot\partial^2_{jk}F \tag{3}$$

$$\partial^3_{jli}\phi^k + \partial^3_{kli}\phi^j + \partial^2_{jl}\hat{F}\cdot\partial^2_{ki}\hat{F} + \partial^2_{ji}\hat{F}\cdot\partial^2_{kl}\hat{F} = \partial^2_{jl}F\cdot\partial^2_{ki}F + \partial^2_{ji}F\cdot\partial^2_{kl}F \tag{4}$$

$$\partial^3_{kij}\phi^l + \partial^3_{lij}\phi^k + \partial^2_{ki}\hat{F}\cdot\partial^2_{lj}\hat{F} + \partial^2_{kj}\hat{F}\cdot\partial^2_{li}\hat{F} = \partial^2_{ki}F\cdot\partial^2_{lj}F + \partial^2_{kj}F\cdot\partial^2_{li}F \tag{5}$$

$$\partial^3_{ljk}\phi^i + \partial^3_{ijk}\phi^l + \partial^2_{lj}\hat{F}\cdot\partial^2_{ik}\hat{F} + \partial^2_{lk}\hat{F}\cdot\partial^2_{ij}\hat{F} = \partial^2_{lj}F\cdot\partial^2_{ik}F + \partial^2_{lk}F\cdot\partial^2_{ij}F. \tag{6}$$

Therefore, the equation

$$(3) - (4) + (5) - (6) \tag{7}$$

eliminates ϕ and gives

$$\partial^2_{ik}\hat{F}\cdot\partial^2_{jl}\hat{F} - \partial^2_{il}\hat{F}\cdot\partial^2_{ik}\hat{F} = \partial^2_{ik}F\cdot\partial^2_{jl}F - \partial^2_{il}F\cdot\partial^2_{ik}F, \tag{8}$$

It follows that, if H and $\widehat{H}$ are the second fundamental forms of M and $\widehat{M}$ at 0, then

$$\widehat{R}_{ijkl} = R_{ijkl}, \tag{9}$$

where

$$R_{ijkl} = H_{ik}\cdot H_{jl} - H_{il}\cdot H_{jk} \tag{10}$$

$$\widehat{R}_{ijkl} = \widehat{H}_{ik}\cdot\widehat{H}_{jl} - \widehat{H}_{il}\cdot\widehat{H}_{jk}. \tag{11}$$

Since H is a tensor at 0, R is also a tensor, where for any $v_1, v_2, v_3, v_4 \in T_pM$,

$$R(v_1, v_2, v_3, v_4) = H(v_1, v_3)\cdot H(v_2, v_4) - H(v_1, v_4)\cdot H(v_2, v_3).$$

Equation 9 shows that the tensor R does not depend on the isometric embedding of M and therefore is an intrinsic geometric invariant. The tensor R is, of course, the Riemann curvature tensor, and the equations (10) are the Gauss equations.

If $M = 2$ and $n = 3$, then the only nontrivial component of the Riemann curvature tensor is

$$K = R(e_1, e_2, e_1, e_2) = H_{11}H_{22} - H_{12}^2,$$

which is known as the Gauss curvature, and equation (9) is the Theorem Egregium of Gauss.

3. Tensor identities

Most of the proof above involves only differentiation and straightforward calculations. The only significant steps are the following:

(1) Definition of a submanifold as a graph over the tangent plane at a point
(2) Definition of the second fundamental form as a Hessian
(3) If a map between two submanifolds preserves lengths of curves, then the map satisfies equation (1).
(4) Most importantly, the tensor calculations done in (2) and (7). These are equivalent to the following tensor identities:

$$(T\otimes S^2T)\cap(\Lambda^2T\otimes T) = \{0\}$$
$$(T\otimes S^3T)\cap(\Lambda^2T\otimes\Lambda^2T) = \{0\}.$$

4. Intrinsic formula for Riemann curvature

The Riemannian metric at $x \in M$ is given by

$$\begin{aligned} g_{ij}(x) &= \partial_i(x, F(x)) \cdot \partial_j(x, F(x)) \\ &= (e_i, \partial_i F) \cdot (e_j, \partial_j F(v)) \\ &= \delta_{ij} + \partial_i F \partial_j F. \end{aligned}$$

Since $dF(0) = 0$, it follows that at $x = 0$,

$$\begin{aligned} \partial_k g_{ij} &= \partial_k(\delta_{ij} + \partial_i F \cdot \partial_j F) \\ &= \partial^2_{ki} F \cdot \partial_j F + \partial_i F \cdot \partial^2_{kj} F \\ &= 0. \\ \partial^2_{ij} g_{kl} &= \partial^2_{ij}(\partial_k F \cdot \partial_l F) \\ &= \partial^2_{ik} F \cdot \partial^2_{jl} F + \partial^2_{il} F \cdot \partial^2_{jk} F. \end{aligned}$$

Permuting the indices, we get

$$\partial^2_{ik} g_{jl} = \partial^2_{ij} F \cdot \partial^2_{kl} F + \partial^2_{il} F \cdot \partial^2_{jk} F \tag{12}$$

$$\partial^2_{jl} g_{ik} = \partial^2_{ij} F \cdot \partial^2_{kl} F + \partial^2_{jk} F \cdot \partial^2_{il} F \tag{13}$$

$$\partial^2_{il} g_{jk} = \partial^2_{ij} F \cdot \partial^2_{kl} F + \partial^2_{ik} F \cdot \partial^2_{jl} F \tag{14}$$

$$\partial^2_{jk} g_{il} = \partial^2_{ij} F \cdot \partial^2_{kl} F + \partial^2_{jl} F \cdot \partial^2_{ik} F. \tag{15}$$

Therefore, the equation

$$-(12) - (13) + (14) + (15) \tag{16}$$

implies that, at $x = 0$,

$$\begin{aligned} -\partial^2_{ik} g_{jl} - \partial^2_{jl} g_{ik} + \partial^2_{il} g_{jk} + \partial^2_{jk} g_{il} &= -\partial^2_{il} F \cdot \partial^2_{jk} F - \partial^2_{jk} F \cdot \partial^2_{jl} F + \partial^2_{ik} F \cdot \partial^2_{jl} F + \partial^2_{jl} F \cdot \partial^2_{ik} F \\ &= 2R_{ijkl}. \end{aligned}$$

This also proves, at least for a Riemannian manifold that can be isometrically embedded as a submanifold in Euclidean space, that the Riemann curvature tensor is an intrinsic geometric invariant.

5. The Riemann curvature of an abstract Riemannian manifold

Let M be a m-manifold with Riemannian metric g. Given local coordinates $x^1, \dots, x^m$, the metric is

$$g = g_{ij}(x)\, dx^i\, dx^j,$$

where $g_{ij}(x) = g(\partial_i, \partial_j)$. The calculations in the previous section suggest the following: Given $p \in M$, let $x = (x^1, \dots, x^m)$ be coordinates such that $x(p) = 0$, $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$, for all $1 \leq i, j, k \leq n$. Let R be the tensor such that for any $v_1, v_2, v_3, v_4 \in T_pM$,

$$R(v_1, v_2, v_3, v_4) = R_{ijkl} v_1^i v_2^j v_3^k v_4^l,$$

where

$$R_{ijkl} = -\frac{1}{2}(\partial^2_{ik} g_{jl} + \partial^2_{jl} g_{ik} - \partial^2_{il} g_{jk} - \partial^2_{jk} g_{il}).$$

To prove that this is a well defined coordinate-independent tensor on M, it suffices to prove the following two lemmas.

Lemma 1. *Let g be a smooth Riemannian metric on an n-manifold M. For each $p \in M$, there exist local coordinates $x = (x^1, \dots, x^m)$ such that $x(p) = 0$ and $g = g_{ij}\, dx^i\, dx^j$, where, for every $1 \le i, j, k \le n$,*

$$\begin{aligned} g_{ij}(0) &= \delta_{ij} \\ \partial_k g_{ij}(0) &= 0. \end{aligned}$$

Remark. Exponential or normal coordinates satisfy Lemma 1. However, a direct proof of Lemma 1 is a lot simpler than the construction of exponential coordinates.

The second lemma shows that the tensor R is independent of coordinates.

Lemma 2. *Let M and N be smooth n-manifolds, h be a Riemannian metric on N, $\Phi : M \to N$ a smooth map, and $g = \Phi^* h$. Let $x = (x^1, \dots, x^m)$ be local coordinates on a neighborhood of $p \in M$ and $y = (y^1, \dots, y^m)$ be local coordinates on a neighborhood of $\Phi(p) \in N$ such that*

$$\begin{aligned} x(p) &= 0 \\ y(\Phi(p)) &= 0 \\ \frac{\partial y^i}{\partial x^j}(0) &= \delta^i_j \\ h_{ij}(0) &= \delta_{ij} \\ \partial_k h_{ij}(0) &= 0. \end{aligned}$$

Then at $x = y = 0$,

$$\begin{aligned} g_{ij}(0) &= \delta_{ij} \\ \partial_k g_{ij}(0) &= 0 \\ \partial^2_{ik} g_{jl} + \partial^2_{jl} g_{ik} - \partial^2_{il} g_{jk} - \partial^2_{jk} g_{il} &= \partial^2_{ik} h_{jl} + \partial^2_{jl} h_{ik} - \partial^2_{il} h_{jk} - \partial^2_{jk} h_{il}. \end{aligned}$$

The proofs of these lemmas can be found in *A simple way to discover the Riemann curvature tensor*.

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