

A GENERALIZED AFFINE ISOPERIMETRIC INEQUALITY

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ABSTRACT. A purely analytic proof is given for an inequality that has as a direct consequence the two most important affine isoperimetric inequalities of plane convex geometry: The Blaschke-Santaló inequality and the affine isoperimetric inequality of affine differential geometry.

1. INTRODUCTION.

In [3], Harrell showed how an analytic approach could be used to obtain a well-known *Euclidean* inequality of plane convex geometry – the Blaschke-Lebesgue inequality. In this article we show how a purely analytic approach can be used to establish the best known *affine* inequalities of plane convex geometry. To be precise, we will use a purely analytic approach to establish an analytic inequality that has as an immediate consequence both the affine isoperimetric inequality of affine differential geometry and the Blaschke-Santaló inequality. What’s more significant, we are able to remove the “convexity” assumption and thus establish an inequality with applications to the planar L_p Minkowski problem (see, e.g., [9], [10], [15], [16], [17]) with not necessarily positive data.

Let $C \subset \mathbf{R}^2$ be a compact convex set. Let \mathbf{S} be the unit circle parameterized by

$$e(\theta) := (\cos \theta, \sin \theta).$$

Then $h = h_C : \mathbf{S} \rightarrow \mathbf{R}$ defined by

$$h(\theta) := \max_{x \in C} e(\theta) \cdot x$$

is the *support function* of C .

The affine isoperimetric inequality of affine differential geometry states that if a plane convex figure has support function $h \in C^2(\mathbf{S})$, then

$$(1.1) \quad 4\pi^2 \int_{\mathbf{S}} h(h + h'') d\theta \geq \left(\int_{\mathbf{S}} (h + h'')^{2/3} d\theta \right)^3$$

with equality if and only if the figure is an ellipse.

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The integral on the left is twice the area of the figure, while the integral on the right is the so called *affine perimeter* of the figure.

The Blaschke-Santaló inequality states that if a convex figure is *positioned* so that its support function h is positive and

$$(1.2) \quad \int_{\mathbf{S}} \frac{\sin \theta}{h(\theta)^3} d\theta = 0 = \int_{\mathbf{S}} \frac{\cos \theta}{h(\theta)^3} d\theta,$$

then

$$(1.3) \quad 4\pi^2 \left(\int_{\mathbf{S}} h^{-2} d\theta \right)^{-1} \geq \int_{\mathbf{S}} h(h + h'') d\theta,$$

with equality if and only if the figure is an ellipse.

The integral on the left is twice the area of the *polar reciprocal* of the figure.

In [7], it was shown that both inequalities (1.1) and (1.3) are encoded in the following inequality: If K and L are convex figures whose support functions are such that $h_L \in C^2(\mathbf{S})$ and h_K arbitrary then

$$(1.4) \quad 4\pi^2 \left(\int_{\mathbf{S}} (h_L + h_L'') h_K d\theta \right)^2 \geq \left(\int_{\mathbf{S}} (h_L + h_L'')^{2/3} d\theta \right)^3 \int_{\mathbf{S}} [h_K^2 - (h_K')^2] d\theta$$

with equality if and only if K and L are homothetic ellipsoids.

Using this version of the inequality, we see that choosing $L = K$ in (1.4) immediately gives (1.1). To see how (1.4) gives (1.3) choose the figure L so that h_L satisfies the equation $h_L'' + h_L = h_K^{-3}$.

In [7], it was shown that (1.4) is a consequence of (1.1) and the *mixed area* inequality. The aim of this paper is to establish an analytic inequality that extends inequality (1.4). Our proof of this new analytic inequality uses none of the tools of convex geometry. We are thus able to remove the “convexity” assumption, $h'' + h \geq 0$, from all the inequalities above.

2. THE MAIN INEQUALITY.

Let $H^1(\mathbf{S})$ be the Hilbert space of functions $u: \mathbf{S} \rightarrow \mathbf{R}$ with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbf{S}} [u^2 + (u')^2] d\theta \right)^{\frac{1}{2}}.$$

Theorem 1 (Two Dimensional Analytic Affine Isoperimetric Inequality).

Assume

i) F and h are non-negative 2π periodic functions that do not vanish identically.

ii) F is integrable on S and satisfies the orthogonality conditions

$$(2.1) \quad \int_{\mathbf{S}} F(\theta) \cos \theta d\theta = 0 = \int_{\mathbf{S}} F(\theta) \sin \theta d\theta.$$

iii) $h \in H^1(\mathbf{S})$.

Then

$$(2.2) \quad \left(\int_{\mathbf{S}} F(\theta) h(\theta) d\theta \right)^2 \geq \frac{1}{4\pi^2} \left(\int_{\mathbf{S}} F^{2/3} d\theta \right)^3 \left(\int_{\mathbf{S}} [h^2 - (h')^2] d\theta \right).$$

Equality holds if and only if there exist $k_1, k_2, a > 0$, and $\alpha \in \mathbf{R}$ such that

$$(2.3) \quad h(\theta) = k_1 \sqrt{a^2 \cos^2(\theta - \alpha) + a^{-2} \sin^2(\theta - \alpha)}$$

and F is given almost everywhere by

$$(2.4) \quad F(\theta) = k_2 (a^2 \cos^2(\theta - \alpha) + a^{-2} \sin^2(\theta - \alpha))^{-3/2}.$$

Remark 2.1. The functions $h(\theta)$ of the form (2.3) are exactly support functions of the ellipses centered at the origin.

The main ingredient in the proof of Theorem 1 is a family of transforms, that leave a few key integrals invariant and which let us construct maximizing sequences. We will introduce the transforms and study their properties in Section 5. In Section 6, we prove the inequality. In Sections 3 we study some regularity results for support functions of planar convex sets. These regularity results are used in Section 4 to derive the affine isoperimetric inequality for general planar sets.

3. FUNCTION SPACES ASSOCIATED WITH THE INEQUALITY.

Because of the integral $\int_{\mathbf{S}} [h^2 - (h')^2] d\theta$ that appears in Theorem 1, the natural function space for the functions h in the theorem is $H^1(\mathbf{S})$. Moreover in the geometric applications Theorem 1 a natural choice for the function h is to be a support function of a bounded convex set and $H^1(\mathbf{S})$ contains all the support functions. The following characterizes the support functions of bounded convex sets.

Proposition 3.1. *A continuous function $h: \mathbf{S} \rightarrow \mathbf{R}$ is the support function of a bounded convex set if and only if the second distributional derivative h'' satisfies $(h'' + h) \geq 0$ as a distribution.*

The proof of this proposition is elementary, and is left to the reader.

We now describe the smallest function space that contains the support functions of convex sets. Let \mathcal{D} be the set of 2π periodic functions u such that the distributional derivative u'' is a signed measure. The total variation $\|\mu\|_{TV}$ of a signed measure μ on \mathbf{S} is its norm as a linear functional on $C(\mathbf{S})$. That is

$$\|\mu\|_{TV} := \sup \left\{ \int_{\mathbf{S}} \phi(\theta) d\mu(\theta) : \phi \in C(\mathbf{S}), |\phi(\theta)| \leq 1 \right\}.$$

The standard norm on \mathcal{D} is $\|h\|_{L^\infty} + \|h''\|_{TV}$, but, for geometric reasons, we use the equivalent norm

$$\|h\|_{\mathcal{D}} := \|h\|_{L^\infty} + \|h'' + h\|_{TV}.$$

The space \mathcal{D} can also be defined as the functions h on \mathbf{S} that are absolutely continuous and such that the first derivative h' is of bounded variation. As functions of bounded variation are bounded this implies all elements of \mathcal{D} are Lipschitz. Therefore the imbedding $\mathcal{D} \subset C^\alpha(\mathbf{S})$ is compact for $\alpha \in [0, 1)$, where $C^\alpha(\mathbf{S})$ is the space of Hölder continuous functions

Theorem 3.2. *The space $\mathcal{D}(\mathbf{S})$ contains all the support functions of bounded convex sets. Moreover every element of $\mathcal{D}(\mathbf{S})$ is a difference of two support functions. Thus $\mathcal{D}(\mathbf{S})$ is the smallest function space containing all the support functions. More precisely if $f \in \mathcal{D}$ there are support functions $h_1, h_2 \in \mathcal{D}$ with $f = h_1 - h_2$ and*

$$(3.1) \quad \|h_1'' + h_1\|_{TV}, \|h_2'' + h_2\|_{TV} \leq 3\|f'' + f\|_{TV}.$$

Proof. We have already seen that \mathcal{D} contains all the support functions of bounded convex sets. Let $f \in \mathcal{D}(\mathbf{S})$. Then $f'' + f$ is a signed measure. We now claim that we can write $f'' + f = \mu_+ - \mu_-$ where μ_+ and μ_- are non-negative measures with the extra conditions that

$$(3.2) \quad \int_{\mathbf{S}} \cos \theta d\mu_+ = \int_{\mathbf{S}} \cos \theta d\mu_- = \int_{\mathbf{S}} \sin \theta d\mu_+ = \int_{\mathbf{S}} \sin \theta d\mu_- = 0$$

and

$$(3.3) \quad \|\mu_+\|_{TV}, \|\mu_-\|_{TV} \leq 3\|f'' + f\|_{TV}.$$

To start let $f'' + f = \nu_+ - \nu_-$ be the Jordan decomposition (cf. [12, p. 274]) of $f'' + f$. Then ν_+ and ν_- are non-negative measures and $\|f'' + f\|_{TV} = \|\nu_+\|_{TV} + \|\nu_-\|_{TV}$. From the definition of the second distributional derivative (which is formally just integration by parts)

$$\int_{\mathbf{S}} (f'' + f) \cos \theta d\theta = - \int_{\mathbf{S}} f \cos \theta d\theta + \int_{\mathbf{S}} f \cos \theta d\theta = 0$$

and likewise $\int_{\mathbf{S}} (f'' + f) \sin \theta d\theta = 0$. Using this in $f'' + f = \nu_+ - \nu_-$ gives

$$\int_{\mathbf{S}} \cos \theta d\nu_+ = \int_{\mathbf{S}} \cos \theta d\nu_-, \quad \int_{\mathbf{S}} \sin \theta d\nu_+ = \int_{\mathbf{S}} \sin \theta d\nu_-.$$

Set

$$a := \frac{1}{\pi} \int_{\mathbf{S}} \cos \theta d\nu_+ = \frac{1}{\pi} \int_{\mathbf{S}} \cos \theta d\nu_-,$$

$$b := \frac{1}{\pi} \int_{\mathbf{S}} \sin \theta d\nu_+ = \frac{1}{\pi} \int_{\mathbf{S}} \sin \theta d\nu_-.$$

Let $C > 0$, to be chosen shortly, and set

$$\mu_+ = \nu_+ + (C - a \cos \theta - b \sin \theta) d\theta$$

$$\mu_- = \nu_- + (C - a \cos \theta - b \sin \theta) d\theta.$$

There is an α so that $a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \cos(\theta + \alpha)$. Thus if $C := \sqrt{a^2 + b^2}$ the measures μ_+ and μ_- are non-negative. Using that in

L^2 the function $\cos \theta$ is orthogonal to $\sin \theta$ and to the constants and that $\int_{\mathbf{S}} \cos^2 \theta d\theta = \pi$

$$\begin{aligned} \int_{\mathbf{S}} \cos \theta d\mu_+ &= \int_{\mathbf{S}} \cos \theta d\nu_+ + \int_{\mathbf{S}} \cos \theta (C - a \cos \theta - b \sin \theta) d\theta \\ &= \int_{\mathbf{S}} \cos \theta d\nu_+ - a \int_{\mathbf{S}} \cos^2 \theta d\theta = 0 \end{aligned}$$

and likewise all the other conditions of (3.2) hold. As $\|f'' + f\|_{TV} = \|\nu_+\|_{TV} + \|\nu_-\|_{TV}$ we have $\min\{\|\nu_+\|_{TV}, \|\nu_-\|_{TV}\} \leq \frac{1}{2}\|f'' + f\|_{TV}$. The formulas for a imply

$$\begin{aligned} |a| &\leq \min \left\{ \frac{1}{\pi} \int_{\mathbf{S}} |\cos \theta| d\nu_+(\theta), \frac{1}{\pi} \int_{\mathbf{S}} |\cos \theta| d\nu_-(\theta) \right\} \\ &\leq \frac{1}{\pi} \min \{ \|\nu_+\|_{TV}, \|\nu_-\|_{TV} \} \leq \frac{1}{2\pi} \|f'' + f\|_{TV}. \end{aligned}$$

Likewise $|b| \leq (2\pi)^{-1}\|f'' + f\|_{TV}$. Using this in the definition of C gives $C \leq \sqrt{2}(2\pi)^{-1}\|f'' + f\|_{TV}$. As μ_+ and μ_- are non-negative measures their total variation is just their total mass. Thus

$$\begin{aligned} \|\mu_{\pm}\|_{TV} &= \int_{\mathbf{S}} 1 d\mu_{\pm} = \int_{\mathbf{S}} 1 \nu_{\pm} + \int_{\mathbf{S}} (C - a \cos \theta - b \sin \theta) d\theta \\ &= \|\nu_{\pm}\| + 2\pi C \leq (1 + \sqrt{2})\|f'' + f\|_{TV} \\ &\leq 3\|f'' + f\|_{TV} \end{aligned}$$

This shows that (3.3) holds.

We claim that there is a function h_+ so that $h_+'' + h_+ = \mu_+$. To see this expand μ_+ in a Fourier series and use the equations (3.2) to see that the coefficients of \sin and \cos vanish.

$$\mu_+ = \frac{a_0}{2} + \sum_{k=2}^{\infty} (a_k \cos(k\theta) + b_k \sin(k\theta)).$$

Then h_+ is given explicitly by

$$h_+(\theta) = \frac{a_0}{2} + \sum_{k=2}^{\infty} \frac{a_k \cos(k\theta) + b_k \sin(k\theta)}{1 - k^2}.$$

The formulas $a_k = \pi^{-1} \int_{\mathbf{S}} \cos(k\theta) d\mu_+(\theta)$, $b_k = \pi^{-1} \int_{\mathbf{S}} \sin(k\theta) d\mu_+(\theta)$ imply that $|a_k|, |b_k| \leq \frac{1}{2}\|\mu_+\|_{TV}$. Therefore the series defining h_+ converges uniformly and thus h_+ is continuous. Likewise there is a continuous function h_- with $h_-'' + h_- = \mu_-$. As μ_+ and μ_- are non-negative measures and formal differentiation of Fourier series corresponds to taking distributional derivatives, both h_+ and h_- are support functions.

Let $y = f - (h_+ - h_-)$. Then $y'' + y = 0$. This implies $y = \alpha \cos \theta + \beta \sin \theta$ for some constants α and β . Thus $f = (h_+ + \alpha \cos \theta + \beta \sin \theta) - h_-$. But $\alpha \cos \theta + \beta \sin \theta$ is the support function of the point (α, β) . So $(h_+ + \alpha \cos \theta + \beta \sin \theta)$ is a support function, and f is a difference of support functions as

required. Letting $h_1 = h_+ + \alpha \cos \theta + \beta \sin \theta$ and $h_2 = h_-$ then $f = h_1 - h_2$ and for $i = 1, 2$ and by (3.3) $\|h_i'' + h_i\|_{TV} = \|\mu_{\pm}\| \leq 3\|f'' + f\|_{TV}$. \square

4. THE AFFINE ISOPERIMETRIC INEQUALITY FOR ARBITRARY PLANAR CONVEX SETS.

If h is a support function, then h is Lipschitz and therefore absolutely continuous. Therefore the distributional derivative h' of h is just the classical derivative which exists almost everywhere. As h is a support function then by Proposition 3.1 the second distributional derivative h'' is a measure and therefore h' is of bounded variation. By a theorem of Lebesgue, the function h' will be differentiable (in the classical sense) almost everywhere. Denote this derivative of h' by Dh' to distinguish it from the distributional derivative. In what follows we will denote classical derivatives of a function f by Df . As the first distributional derivative of h agrees with the classical derivative we have $Dh' = D^2h$ so that Dh' is the second classical derivative.

Recall that, by a theorem of Alexandrov, a convex function on an n dimensional space, and thus a support function, has a generalized second derivative, called the Alexandrov second derivative, almost everywhere and in the one dimensional case the Alexandrov second derivative is just D^2h .

Various authors [6, 14, 8, 18] have extended the definition of affine ar-length (and more generally higher dimensional affine surface area) from convex sets with C^2 boundary to general convex sets. It was eventually that shown all these definitions are equivalent see i.e. [1] and, for two dimensional convex sets, are given in terms of the support function by

$$\int_{\mathbf{S}} (D^2h + h)^{2/3} d\theta.$$

The following is the general form of the affine isoperimetric inequality in the plane.

Theorem 4.1. *Let K be an compact convex body in the plane with area A and affine perimeter Ω . Then*

$$(4.1) \quad \Omega \leq 8\pi^2 A$$

with equality if and only if K is an ellipse.

The proof of this Theorem is based on our Theorem 1 and the following result which compares the distributional and classical derivatives of a support function.

Proposition 4.2. *Let $h: \mathbf{S} \rightarrow \mathbf{R}$ be the support function of a bounded convex set. Then the distribution $h'' + h$ is of the form*

$$(4.2) \quad h'' + h = (D^2h + h) d\theta + d\mu$$

where $d\theta$ is Lebesgue measure, the function $D^2h + h$ is in $L^1(\mathbf{S})$ and μ is a non-negative measure that is singular with respect to Lebesgue measure (i.e. there is a set N of Lebesgue measure zero with $\mu(S \setminus N) = 0$).

The proof of the Proposition is elementary and is left to the reader.

Proof of Theorem 4.1. Let h_o be the support function of a planar convex body K . In the proof of Lemma 6.3 we will see that it is possible to choose a_o and b_o so that $h(\theta) := h_o(\theta) + a_o \cos \theta + b_o \sin \theta$ is positive on \mathbf{S} and

$$(4.3) \quad \int_{\mathbf{S}} \frac{\cos \theta}{h^3(\theta)} d\theta = \int_{\mathbf{S}} \frac{\sin \theta}{h^3(\theta)} d\theta = 0.$$

Using the relation between $D^2h + h$ and $h'' + h$ given by Proposition 4.2 we have

$$\int_{\mathbf{S}} h(D^2h + h) d\theta \leq \int_{\mathbf{S}} h(h'' + h) d\theta.$$

This observation, preceded by Hölder's inequality, gives,

$$(4.4) \quad \left(\int_{\mathbf{S}} (D^2h + h)^{2/3} d\theta \right)^3 \leq \left(\int_{\mathbf{S}} \frac{d\theta}{h^2} \right) \left(\int_{\mathbf{S}} h(h'' + h) d\theta \right)^2.$$

In Theorem 1 take $F = h^{-3}$. Then (4.3) shows that conditions (2.1) are satisfied. Therefore

$$(4.5) \quad \begin{aligned} \left(\int_{\mathbf{S}} \frac{d\theta}{h^2} \right)^3 \left(\int_{\mathbf{S}} h(h'' + h) d\theta \right) &= \left(\int_{\mathbf{S}} \frac{d\theta}{h^2} \right)^3 \left(\int_{\mathbf{S}} [h^2 - (h')^2] d\theta \right) \\ &\leq 4\pi^2 \left(\int_{\mathbf{S}} \frac{d\theta}{h^2} \right)^2. \end{aligned}$$

Combining (4.4) and (4.5) and using the fact that $D^2h + h = D^2h_o + h_o$ and $\int_{\mathbf{S}} h(h'' + h) d\theta = \int_{\mathbf{S}} h_o(h''_o + h_o) d\theta$ gives

$$\left(\int_{\mathbf{S}} (D^2h_o + h_o)^{2/3} d\theta \right)^3 \leq 4\pi^2 \int_{\mathbf{S}} h_o(h''_o + h_o) d\theta.$$

This is the affine isoperimetric inequality for K .

If equality holds, then the equality conditions of Theorem 1 imply h is the support function of an ellipse centered at the origin. Thus $h_o = h - a \cos \theta - b \sin \theta$ is the support function of an ellipse centered at $(-a, -b)$. \square

5. A FAMILY OF TRANSFORMS.

Let \mathbf{S} be the unit circle in \mathbf{R}^2 with coordinate θ as above. For each $\lambda \in (0, \infty)$, let

$$\psi_\lambda(\theta) = \sqrt{\lambda^2 \cos^2 \theta + \frac{1}{\lambda^2} \sin^2 \theta}.$$

Define on \mathbf{S} a family of mappings

$$m_\lambda(\theta) = \int_0^\theta \frac{dt}{\psi_\lambda^2(t)}.$$

When $\lambda = 1$, this is the identity map. For $0 \leq \theta < \frac{\pi}{2}$ it is easy to verify that

$$(5.1) \quad m_\lambda(\theta) = \arctan \left(\frac{1}{\lambda^2} \tan \theta \right).$$

For any measurable function u on \mathbf{S} , define the transform

$$(T_\lambda u)(\theta) = u(m_\lambda(\theta))\psi_\lambda(\theta).$$

Lemma 5.1. *Let u and v be measurable functions on \mathbf{S} for which the integrals below exist. Then*

(i) *The mappings $m_\lambda(\cdot)$ each leave four points fixed:*

$$m_\lambda(0) = 0, \quad m_\lambda\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \quad m_\lambda(\pi) = \pi, \quad m_\lambda\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2}.$$

(ii) *The transforms leave the following integrals invariant:*

$$(5.2) \quad \int \frac{d\theta}{(T_\lambda u)^2} = \int \frac{d\theta}{u^2},$$

$$(5.3) \quad \int \frac{T_\lambda u}{(T_\lambda v)^3} d\theta = \int \frac{u}{v^3} d\theta,$$

$$(5.4) \quad \int_{\mathbf{S}} \{(T_\lambda u)^2 - [(T_\lambda u)']^2\} d\theta = \int_{\mathbf{S}} [u^2 - (u')^2] d\theta,$$

$$(5.5) \quad \int \frac{\cos \theta}{(T_\lambda u)^3(\theta)} d\theta = \frac{1}{\lambda} \int \frac{\cos \theta}{u^3(\theta)} d\theta,$$

$$(5.6) \quad \int \frac{\sin \theta}{(T_\lambda u)^3(\theta)} d\theta = \lambda \int \frac{\sin \theta}{u^3(\theta)} d\theta.$$

Here “ \int ” represents the integral with respect to $d\theta$ on any of the intervals $[0, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \pi]$, $[\pi, \frac{3\pi}{2}]$, $[\frac{3\pi}{2}, 2\pi]$, or $[0, 2\pi]$. For (5.2), (5.3), (5.5), and (5.6) u and v can be any measurable functions for which the integrals converge. In (5.4) $u \in H^1(\mathbf{S})$.

Remark 5.2. We indicate briefly the geometric origin of these transformations and the invariance properties of some the integrals in the lemma. Let K be a convex body in \mathbf{R}^2 and let K_λ be the image of K under the linear map $(x, y) \mapsto (\lambda x, \frac{1}{\lambda} y)$. Then the support functions are related by $h_{K_\lambda}(\theta) = (T_\lambda h_K)(\theta)$. Therefore integrals of functions of the support function that represent quantities of the body that are invariant under area preserving affine maps will be invariant under the transformations T_λ . For example when u is a support function (5.4) corresponds to the invariance of the area and (5.5) and (5.6) correspond to the invariance of the Santaló point. The lemma shows these invariance properties extend to a larger class of functions than support functions.

Proof. (i) From (5.1), one can see that $m_\lambda(0) = 0$, $m_\lambda(\frac{\pi}{2}) = \frac{\pi}{2}$. Since the integrand is symmetric about $\theta = \frac{\pi}{2}$ on $[0, \pi]$, and is π -periodic, it follows that π and $\frac{3\pi}{2}$ are also fixed points of m_λ .

(ii) We verify invariance for the integrals on the interval $[0, \frac{\pi}{2}]$. Then by the results in (i) and the symmetry of $\psi_\lambda(\theta)$, $\cos \theta$, and $\sin \theta$, the invariance of the integrals on the other intervals follows.

Equations (5.2) and (5.3) are direct consequences from the substitution $\tilde{\theta} = m_\lambda(\theta)$.

As $C^2(\mathbf{S})$ is dense in $H^1(\mathbf{S})$ it is enough to verify (5.4) in the case $u \in C^2(\mathbf{S})$. After integrating by parts, we only need to show

$$\int_{\mathbf{S}} T_\lambda u [T_\lambda u + (T_\lambda u)'] d\theta = \int_{\mathbf{S}} u(u + u') d\theta.$$

We employ the fact that ψ_λ is a solution of the equation

$$(5.7) \quad \psi_\lambda'' + \psi_\lambda = \frac{1}{\psi_\lambda^3}.$$

It follows from (5.7) and a straightforward calculation that

$$T_\lambda u [T_\lambda u + (T_\lambda u)'] = u(m_\lambda(\theta)) [u(m_\lambda(\theta)) + u'(m_\lambda(\theta))] \frac{1}{\psi_\lambda^2(\theta)}.$$

Again using the change of variable $\tilde{\theta} = m_\lambda(\theta)$, we see (5.4) holds.

To obtain (5.5) and (5.6), we write $\tan \theta = \lambda^2 \tan \tilde{\theta}$. It follows that

$$\frac{\cos \theta}{\psi_\lambda(\theta)} = \frac{1}{\sqrt{\lambda^2 + \frac{1}{\lambda^2} \tan^2 \theta}} = \frac{1}{\lambda} \cos \tilde{\theta}, \quad \frac{\sin \theta}{\psi_\lambda(\theta)} = \frac{1}{\sqrt{\frac{\lambda^2}{\tan^2 \theta} + \frac{1}{\lambda}}} = \lambda \sin \tilde{\theta}$$

and another application of the substitution $\tilde{\theta} = m_\lambda(\theta)$ completes the proofs of (5.5) and (5.6). \square

Remark 5.3. Let u be a positive continuous function on \mathbf{S} , then the integral of $\frac{d\theta}{(T_\lambda u)^2}$ is independent of λ . It is not hard to check that as $\lambda \rightarrow \infty$ the mass of $\frac{d\theta}{(T_\lambda u)^2}$ concentrates about the points $\pi/2$ and $3\pi/2$ and when $\lambda \rightarrow 0$ the mass concentrates about 0 and π .

6. PROOF OF THE MAIN INEQUALITY.

6.1. Some lemmas.

Lemma 6.1. *If $\{u_k\}$ is a bounded sequence in $H^1(\mathbf{S})$, then there exists a subsequence (still denoted by $\{u_k\}$) and $u_o \in H^1(\mathbf{S})$, such that $u_k \rightarrow u_o$ in the weak topology of $H^1(\mathbf{S})$,*

$$(6.1) \quad u_k \rightarrow u_o \quad \text{in } C^\beta(\mathbf{S}) \quad \text{for all } \beta < \frac{1}{2}.$$

This implies

$$(6.2) \quad \limsup_{k \rightarrow \infty} \int_{\mathbf{S}} [u_k^2 - (u_k')^2] d\theta \leq \int_{\mathbf{S}} [u_o^2 - (u_o')^2] d\theta.$$

Moreover, if $u_o(\theta_0) = 0$ at some point θ_0 , then for $\delta > 0$

$$(6.3) \quad \int_{\theta_0}^{\theta_0 + \delta} \frac{d\theta}{u_o^2} = \infty \quad \text{and} \quad \int_{\mathbf{S}} \frac{d\theta}{u_k^2} \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Lemma 6.2. *Assume $\{u_k\}$ is a bounded sequence in $H^1(\mathbf{S})$ with $u_k > 0$, $u_k \rightarrow u_o \in H^1(\mathbf{S})$ in the weak topology,*

$$(6.4) \quad \int_{\mathbf{S}} \frac{\cos \theta}{u_k^3} d\theta = 0 = \int_{\mathbf{S}} \frac{\sin \theta}{u_k^3} d\theta,$$

and that u_o has at least one zero. Then, viewing the zeros of u_o as a subset of $\mathbf{S} \subset \mathbf{R}^2$,

$$(6.5) \quad (0, 0) \in \text{convex hull of the zeros of } u_o.$$

If u_o has three or more zeros then

$$(6.6) \quad \int_{\mathbf{S}} [u_k^2 - (u'_k)^2] d\theta < 0 \quad \text{for sufficiently large } k.$$

Lemma 6.3. *Suppose the inequality (2.2) holds under the stronger conditions:*

- i) F is measurable and positive on \mathbf{S} and $h \in H^1(\mathbf{S})$ is positive;*
- ii) F satisfies the orthogonality conditions (2.1) and h satisfies orthogonality conditions*

$$(6.7) \quad \int_{\mathbf{S}} \frac{\cos \theta}{h^3} d\theta = 0 = \int_{\mathbf{S}} \frac{\sin \theta}{h^3} d\theta.$$

Then the same inequality (2.2) holds without the orthogonality conditions (6.7) on h and the strict positivity of F .

Proof of Lemma 6.1. That there is a $u_o \in H^1(\mathbf{S})$ and a subsequence with $u_k \rightarrow u_o$ in the weak topology follows from the weak compactness of the closed balls in a Hilbert space. Then (6.1) is a direct consequence of the compact Sobolev imbedding of $H^1(\mathbf{S})$ into $C^\beta(\mathbf{S})$ for any $\beta < \frac{1}{2}$. To prove (6.2) use the fact that the norm of a Hilbert space is lower semi-continuous with respect to weak convergence and thus $\liminf_{k \rightarrow \infty} \int_{\mathbf{S}} [u'_k]^2 d\theta \geq \int_{\mathbf{S}} [u'_o]^2 d\theta$. From (6.1) $\lim_{k \rightarrow \infty} \int_{\mathbf{S}} u_k^2 d\theta = \int_{\mathbf{S}} u_o^2 d\theta$. Together these imply (6.2).

Assume that u_o vanishes at θ_0 . Then by the Sobolev imbedding $H^1(\mathbf{S}) \subset C^{\frac{1}{2}}(\mathbf{S})$, or an elementary Hölder inequality argument, $u_o \in C^{\frac{1}{2}}(\mathbf{S})$ and therefore $|u_o(\theta)| = |u_o(\theta) - u_o(\theta_0)| \leq C_1 \sqrt{|\theta - \theta_0|}$ which implies the divergence of the integral $\int_{\theta_0}^{\theta_0 + \delta} u_o^{-2} d\theta$. As $u_k \rightarrow u_o$ uniformly this implies the second part of (6.3) and completes the proof of the Lemma. \square

Proof of Lemma 6.2. Because the imbedding of $H^1(\mathbf{S})$ into $C^\beta(\mathbf{S})$ is compact for $\beta \in [0, 1/2)$ the weak convergence $u_k \rightarrow u_o$ implies $\{u_k\}$ converges to u_o uniformly. By Lemma 6.1 the integral $\int_{\mathbf{S}} u_o^{-2} d\theta$ diverges and therefore $\int_{\mathbf{S}} u_o^{-3} d\theta$ also diverges. Thus $\int_{\mathbf{S}} u_k^{-3} d\theta \rightarrow \infty$ as $k \rightarrow \infty$. Let $c_k := (\int_{\mathbf{S}} u_k^{-3} d\theta)^{-1}$. Then $c_k u_k^{-3}(\theta) d\theta$ is a probability measure on \mathbf{S} and the conditions (6.4) imply the center of mass of this measure is $(0, 0)$. But as $k \rightarrow \infty$ the masses of the measures $c_k u_k^{-3}(\theta) d\theta$ concentrate at the zeros of u_o . This implies (6.5).

If u_o has three or more zeros, then the convex hull property (6.5) implies there are three zeros $\theta_1, \theta_2, \theta_3$ of u_o such that

(6.8) The zeros $\theta_1, \theta_2, \theta_3$ of u_o are not on an arc of length less than π .

We will show this implies

$$(6.9) \quad \int_{\mathbf{S}} [u_o^2 - (u_o')^2] d\theta < 0.$$

Which, by (6.2) of Lemma 6.1, implies (6.6).

To see (6.9), we write the integral in three parts:

$$(6.10) \quad \int_{\mathbf{S}} [u_o^2 - (u_o')^2] d\theta = \left\{ \int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_3}^{\theta_1} \right\} [u_o^2 - (u_o')^2] d\theta = I_1 + I_2 + I_3.$$

From (6.8), we see that lengths of intervals of integration in (6.10) are all less than or equal to π , and at least two of them are strictly less than π . But, [2, p. 185], if $\theta_{i+1} - \theta_i \leq \pi$, then $u_o(\theta_i) = u_o(\theta_{i+1}) = 0$ implies $\int_{\theta_i}^{\theta_{i+1}} [u_o^2 - (u_o')^2] d\theta < 0$ unless $\theta_{i+1} - \theta_i = \pi$ and $u_o = C \sin(\theta - \theta_i)$ on $[\theta_i, \theta_{i+1}]$. This proves (6.9) and completes the proof of the lemma. \square

Proof of Lemma 6.3. We assume that the inequality (2.2) holds under the assumptions i) and ii) of Lemma 6.3. We first claim that for each positive function $h \in C^2(\mathbf{S})$, there exists an $h_o(\theta) = a_o \cos \theta + b_o \sin \theta + h(\theta)$ that satisfies the orthogonality conditions (6.7). To see this minimize the function

$$f(a, b) = \int_{\mathbf{S}} \frac{1}{(a \cos \theta + b \sin \theta + h(\theta))^2} d\theta$$

for any real numbers a and b , such that $a \cos \theta + b \sin \theta + h(\theta) > 0$ for all θ . It is obvious that $f(a, b)$ is bounded from below by zero. Let $\{h_k(\theta) = a_k \cos \theta + b_k \sin \theta + h(\theta)\}$ be a minimizing sequence. From $h_k(t) > 0$, one can easily see that $\{a_k\}$ and $\{b_k\}$ are bounded, and hence there exist subsequences converging to some $a_o, b_o \in \mathbf{R}$. Then (a_o, b_o) is a minimizer of f .

Moreover, from Lemma 6.1, we can see that $h_o(\theta) = a_o \cos \theta + b_o \sin \theta + h(\theta) > 0$. (Otherwise h_o has a zero and (by Lemma 6.1) $\int_{\mathbf{S}} h_o^{-2} d\theta = \infty$, contradicting that h_o is a minimizer.) Consequently, at (a_o, b_o) , we have $\partial f / \partial a = 0 = \partial f / \partial b$. This implies the orthogonality conditions (6.7) on h_o .

We now show that if inequality (2.2) holds for $h_o = a_o \cos \theta + b_o \sin \theta + h(\theta)$, then it also holds for h . By the orthogonality conditions (2.1) on F .

$$(6.11) \quad \int_{\mathbf{S}} F(\theta) h_o(\theta) d\theta = \int_{\mathbf{S}} F(\theta) h(\theta) d\theta,$$

and if h is of class C^2 we can use integration by parts and the fact that both $\sin \theta$ and $\cos \theta$ are in the kernel of the differential operator $d^2/d\theta^2 + 1$ to get

$$(6.12) \quad \begin{aligned} \int_{\mathbf{S}} [h_o^2 - (h_o')^2] d\theta &= \int_{\mathbf{S}} h_o(h_o'' + h_o) d\theta \\ &= \int_{\mathbf{S}} h(h'' + h) d\theta = \int_{\mathbf{S}} [h^2 - (h')^2] d\theta. \end{aligned}$$

This will also hold for $h \in H^1(\mathbf{S})$ by approximating by C^2 functions. So if (2.2) holds for h_o and F , then (6.11) and (6.12) show it holds for h and F .

To see that inequality (2.2) also holds for non-negative continuous functions F and non-negative $h \in H^1(\mathbf{S})$, we let

$$F_\epsilon = F + \epsilon \quad \text{and} \quad h_\epsilon = h + \epsilon.$$

Then obviously, for each $\epsilon > 0$, both F_ϵ and h_ϵ are positive, and F_ϵ satisfies the orthogonality conditions (2.1). Therefore inequality (2.2) holds for F_ϵ and h_ϵ . Take the limit as $\epsilon \rightarrow 0$ to see that (2.2) is also holds for F and h .

Finally the extensions to F non-negative and measurable follow by approximating F by positive functions satisfying the orthogonality conditions (2.1) and taking limits. \square

6.2. Outline of the Proof. Let

$$G = \left\{ v : v > 0, \int_S \frac{1}{v^3} d\theta < \infty, \int_S \frac{1}{v^3} \cos \theta d\theta = 0 = \int_S \frac{1}{v^3} \sin \theta d\theta \right\}$$

and let

$$\tilde{G} = \{u \in G : u \in H^1(S)\}.$$

Define on $\tilde{G} \times G$,

$$I(u, v) = \frac{\left\{ \int_S \frac{1}{v^2} d\theta \right\}^3 \left\{ \int_S [u^2 - (u')^2] d\theta \right\}}{\left\{ \int_S \frac{u}{v^3} d\theta \right\}^2}.$$

It is obvious that for any constants t and s , we have

$$I(tu, sv) = I(u, v).$$

To prove the theorem, it is equivalent to show that

$$(6.13) \quad I(u, v) \leq 4\pi^2 \quad \forall (u, v) \in \tilde{G} \times G.$$

and the equality holds if and only if

$$(6.14) \quad u = k_1 \sqrt{\lambda^2 \cos^2 \theta + \frac{1}{\lambda^2} \sin^2 \theta}$$

and

$$(6.15) \quad v = k_2 \sqrt{\lambda^2 \cos^2 \theta + \frac{1}{\lambda^2} \sin^2 \theta}$$

with any non-negative constants k_1, k_2 , and $\lambda > 0$.

First, we show that there exists a constant $C < \infty$, such that

$$I(u, v) \leq C \quad \forall (u, v) \in \tilde{G} \times G.$$

It is done by applying the family of transforms and by using a contradiction argument.

Then we study a maximizing sequence $\{(u_k, v_k)\}$ of the functional $I(u, v)$. Usually, such a sequence may be unbounded. However thanks to the family

of transforms, we are able to convert it into a new sequence which converges to a maximum (u_o, v_o) in $\tilde{G} \times G$. Finally, we use the well-known classification results on the solutions of the corresponding Euler-Lagrange equations to arrive at the conclusion of the theorem.

6.3. The Proof. Part I.

In this part, we show that there exists a constant $C < \infty$, such that

$$(6.16) \quad I(u, v) \leq C, \quad \forall (u, v) \in \tilde{G} \times G.$$

We argue by contradiction. Suppose in contrary, there exists a sequence $\{(\tilde{u}_k, v_k)\}$ in $\tilde{G} \times G$, such that $I(\tilde{u}_k, v_k) \rightarrow \infty$, as $k \rightarrow \infty$.

Let $u_k = \frac{\tilde{u}_k}{\|\tilde{u}_k\|_{H^1}}$, then

$$(6.17) \quad \|u_k\|_{H^1} = 1, \quad \text{and } I(u_k, v_k) \rightarrow \infty.$$

Here and in the rest of the paper, for convenience of writing, we use $\{(u_k, v_k)\}$ to denote the sequence itself or one of its subsequences.

The first part of (6.17) implies that

$$(6.18) \quad J(u_k) = \int_{\mathbf{S}} [u_k^2 - (u_k')^2] d\theta \text{ is bounded,}$$

Therefore from the second part, we must have

$$(6.19) \quad u_k(\theta) \rightarrow 0, \quad \text{for some } \theta,$$

since otherwise, if $\{u_k\}$ is bounded away from zero, then by the Hölder inequality

$$(6.20) \quad \int_{\mathbf{S}} \frac{1}{v^2} d\theta \leq \left(\int_{\mathbf{S}} \frac{u}{v^3} d\theta \right)^{2/3} \left(\int_{\mathbf{S}} \frac{1}{u^2} d\theta \right)^{1/3}$$

we would arrive at the boundedness of

$$\left(\int_{\mathbf{S}} \frac{1}{v_k^2} d\theta \right)^3 \left(\int_{\mathbf{S}} \frac{u_k}{v_k^3} d\theta \right)^{-2}.$$

This, together with (6.18), contradicts with the second part of (6.17).

Without loss of generality we assume that $J(u_k) > 0$ for all k . By going to a subsequence we can assume that $\{u_k\}$ has a weak limit u_o . We have shown that u_o has at least one zero and therefore by the convex hull property of Lemma 6.2 the point $(0, 0)$ is in the convex hull of the zeros of u_o . This implies that u_o has at least two zeros. If u_o has three or more zeros then (6.6) of Lemma 6.2 implies that $J(u_k) < 0$ which is not the case. Thus u_o has exactly two zeros.

As u_o has exactly two zeros, the convex hull property (6.5) implies the two zeros must be antipodal, say they are at $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Obviously, at

these two points, $u_k^{-2} \rightarrow \infty$. For each u_k , pick a point p_k near $\pi/2$, such that

$$(6.21) \quad \int_{p_k - \frac{\pi}{2}}^{p_k} \frac{1}{u_k^2} d\theta = \int_{p_k}^{p_k + \frac{\pi}{2}} \frac{1}{u_k^2} d\theta.$$

Then, $p_k \rightarrow \pi/2$. For a number $\delta > 0$ (to be concrete $\delta = \pi/4$ will work), let

$$D_k^1 = \left\{ \theta : p_k - \frac{\pi}{2} \leq \theta \leq p_k - \delta \right\}, \quad D_k^2 = \left\{ \theta : p_k + \delta \leq \theta \leq p_k + \frac{\pi}{2} \right\}$$

$$D_k = D_k^1 \cup D_k^2.$$

We will apply the family of transforms T_λ introduced in Section 5. We say that the family of transforms T_λ in Lemma 6.1 are centered at $\frac{\pi}{2}$, and write $T_\lambda = T_{\lambda, \frac{\pi}{2}}$. Similarly, one can define transforms centered at any point q , and denote them by $T_{\lambda, q}$.

By Lemma 5.1 and Remark 5.3 for each u_k , one can choose a transform $T_k = T_{\lambda_k, p_k}$, such that

$$(6.22) \quad \int_{B_\delta(p_k)} \frac{d\theta}{(T_k u_k)^2} = \int_{D_k} \frac{d\theta}{(T_k u_k)^2},$$

where $B_\delta(p_k) = (p_k - \delta, p_k + \delta)$. Let $w_k = \|T_k u_k\|_{H^1}^{-1} T_k u_k$. Then $\|w_k\|_{H^1} = 1$ and we can apply Lemma 6.1 to the sequence $\{w_k\}$ and find a subsequence, still denoted by $\{w_k\}$, and a $w_o \in H^1(\mathbf{S})$ such that $w_k \rightarrow w_o$ in the weak topology of H^1 and $w_k(\theta) \rightarrow w_o(\theta)$ in C^β for all $\beta < \frac{1}{2}$.

If w_o has no zeros, then by (5.2) in Lemma 5.1 and (6.20), $I(u_k, v_k)$ is bounded, and we are done.

Therefore, we may assume that w_o has at least one zero. By the convex hull property of Lemma 6.2

$$(6.23) \quad (0, 0) \in \text{convex hull of the zeros of } w_o.$$

This implies that w_o has at least two zeros and if w_o has three or more zeros then Lemma 6.2 implies $J(w_k) < 0$ for large k , which, by (5.4) of Lemma 5.1, again would contradict with the assumption that $J(u_k) > 0$. Therefore w_o has exactly two zeros and by the convex hull property (6.23) these zeros are antipodal. Let the zeros be θ_0 and θ_1 and we can assume that $\theta_0 \in [0, \pi]$. Then $\int_0^\pi w_o^{-2} d\theta = \infty$ (by (6.3)), $w_k \rightarrow w_o$ uniformly, and $p_k \rightarrow \pi/2$ imply

$$(6.24) \quad \int_{p_k - \frac{\pi}{2}}^{p_k + \frac{\pi}{2}} \frac{d\theta}{w_k^2} \rightarrow \infty.$$

From (6.21), (6.22) and the properties of the transforms, we also have

$$(6.25) \quad \int_{p_k - \frac{\pi}{2}}^{p_k} \frac{d\theta}{w_k^2} = \int_{p_k}^{p_k + \frac{\pi}{2}} \frac{d\theta}{w_k^2},$$

and

$$(6.26) \quad \int_{B_\delta(p_k)} \frac{d\theta}{w_k^2} = \int_{D_k} \frac{d\theta}{w_k^2}.$$

Now (6.24), (6.25), and (6.26) imply that the integrals of w_k^{-2} on all the four sets

$$[p_k - \frac{\pi}{2}, p_k], [p_k, p_k + \frac{\pi}{2}], B_\delta(p_k), \text{ and } D_k$$

approach infinity. Therefore w_o has at least one zero on each of the following sets

$$(6.27) \quad [0, \frac{\pi}{2}], [\frac{\pi}{2}, \pi], B_\delta(\frac{\pi}{2}), \text{ and } [0, \pi] \setminus B_\delta(\frac{\pi}{2}).$$

From this we see that w_o has at least two zeros on the closed upper half circle. As the two zeros of w_o are antipodal they must be 0 and π . But as $B_\delta(\pi/2)$ also contains a zero this implies that w_o has three zeros, a contradiction.

Part II.

In part I, we have shown that

$$I(u, v) \leq C < \infty \quad \forall (u, v) \in \tilde{G} \times G.$$

To obtain the least possible value of the constant C, we consider a maximizing sequence $\{(\tilde{u}_k, v_k)\}$ with $\|\tilde{u}_k\| = 1$.

Using an entire similar argument as in part I, we can show that there exists a family of transforms $T_k = T_{\lambda_k, p_k}$, such that for $u_k = \frac{T_k \tilde{u}_k}{\|T_k \tilde{u}_k\|}$, we have

$$u_k(\theta) \rightarrow u_o(\theta) > 0, \quad \forall \theta \in [0, 2\pi].$$

It follows from (6.20) that

$$(6.28) \quad I(u_k, v_k) \leq I(u_k, u_k).$$

Thus, (u_o, u_o) is a maximum of $I(u, v)$; and furthermore, it is in the interior of $\tilde{G} \times \tilde{G}$. Therefore, we have

$$\frac{\partial I}{\partial u} \Big|_{\tilde{G} \times \tilde{G}} (u_o, u_o) = 0.$$

Through a straightforward calculation, one can see that a constant multiple of u_o , still denoted by u_o , satisfies the following Euler-Lagrange equation

$$(6.29) \quad u_o''(\theta) + u_o(\theta) = \frac{1}{u_o^3} + a \frac{\cos \theta}{u_o^4} + b \frac{\sin \theta}{u_o^4}.$$

To determine the constants a and b, we multiply both sides of (6.29) by $\cos \theta$ and $\sin \theta$ respectively, then integrate over $[0, 2\pi]$ to obtain

$$(6.30) \quad a \int_{\mathbf{S}} \frac{\cos^2 \theta}{u_o^4} d\theta + b \int_{\mathbf{S}} \frac{\sin \theta \cos \theta}{u_o^4} d\theta = 0,$$

$$(6.31) \quad a \int_{\mathbf{S}} \frac{\cos \theta \sin \theta}{u_o^4} d\theta + b \int_{\mathbf{S}} \frac{\sin^2 \theta}{u_o^4} d\theta = 0.$$

Using the Hölder inequality, one can show that

$$(6.32) \quad \left| \begin{array}{cc} \int_{\mathbf{S}} \frac{\cos^2 \theta}{u_o^4} d\theta & \int_{\mathbf{S}} \frac{\sin \theta \cos \theta}{u_o^4} d\theta \\ \int_{\mathbf{S}} \frac{\sin \theta \cos \theta}{u_o^4} d\theta & \int_{\mathbf{S}} \frac{\sin^2 \theta}{u_o^4} d\theta \end{array} \right| > 0.$$

Therefore the algebraic system (6.30) and (6.31) has only the trivial solution $a = b = 0$. Consequently, u_o satisfies

$$(6.33) \quad u_o'' + u_o = \frac{1}{u_o^3}.$$

Now by the well-known classification result for equation (6.33) (see e.g. [11, p. 120]), we have

$$u_o(\theta) = \sqrt{\lambda^2 \cos^2 \theta + \frac{1}{\lambda^2} \sin^2 \theta}$$

for some constant λ .

Then, a straight forward calculation leads to

$$I(u_o, u_o) = 4\pi^2.$$

Finally, since in (6.20), the equality holds if and only if v is a constant multiple of u , we see that if (u_o, v_o) is a maximum of the functional $I(u, v)$, then v_o must be a constant multiple of u_o .

This completes the proof of the theorem.

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