AFFINE INTEGRAL GEOMETRY FROM A DIFFERENTIABLE VIEWPOINT

DEANE YANG

I dedicate this to the sixtieth birthday of Shing Tung Yau

Abstract. The notion of a homogeneous contour integral is introduced and used to construct affine integral invariants of convex bodies. Earlier descriptions of this construction rely on a Euclidean structure on the ambient vector space, so the behavior under linear transformations is established after the fact. The description presented here relies only on the linear structure and a Lebesgue measure on the ambient vector space, making its behavior under linear transformations more transparent.

Also, recent work by Ludwig classifying different types of affine or linearly invariant valuations on convex bodies is reviewed. The invariants obtained using the homogeneous contour integral are exactly the invariants that arise in Ludwig's classification theorems.

1. Introduction

The subject of *affine geometry* can be described as the study of affine invariant properties of subsets of n-dimensional Euclidean space $\mathbf{R}^n$. Here, *affine invariant* means invariant under the group of *affine transformations*, which is the group generated by linear transformations and translations.

There are two distinct approaches to this subject. One is known as *affine differential geometry*. This approach is analogous to the one taken in Euclidean or Riemannian differential geometry. One studies affine invariant local differential invariants of the set or its boundary. Recent accounts of affine differential geometry include [56, 88–90, 99]. This approach will not be discussed here.

Instead, we will describe here one aspect of *affine integral geometry*, where one studies affine geometric invariants of a convex body in $\mathbf{R}^n$ obtained by integration or averaging. Although this is an active

Date: February 2, 2009, revised March 29, 2009 and April 6, 2009.

2000 *Mathematics Subject Classification.* 52A22,53C65,52B45,52A39,52A20.

Key words and phrases. convex body,affine invariant,valuation,integral geometry,convex geometry,affine geometry.

area of research, it is perhaps not familiar to most differential geometers. There is no shortage of surveys and research monographs on the subject, including books by Schneider [94], Gardner [35], and Thompson [100], as well as an excellent survey article by Gardner [36].

In this short article, we will focus on recent progress [57–63, 65] towards the classification of all possible affine integral invariants of a convex body. These results will be presented from a viewpoint that might be more comfortable and easier to digest for differential geometers but that is not seen in most writings on affine integral geometry. Ludwig [63] has written another and particularly beautiful survey taking a different approach to the same topics. Ludwig-Haberl [42] and Haberl [40, 41] have also established classification theorems with assumptions different from those considered here. Also worth noting is McMullen's classification of translation invariant valuations [81].

Another motivation for this article is to present a new approach, more transparent than other commonly used ones, to constructing affine integral invariants of a convex body. There are two previously known approaches. One, favored by differential geometers and analysts, is to construct appropriate measures, usually involving a generalized notion of curvature, on the boundary of the body and integrate over them. There are, however, significant technical difficulties in working with such measures for a general convex body. The second less technical approach, favored by convex geometric analysts, is to construct appropriate measures on the unit sphere (viewed as the space of all possible outward unit normals to the boundary). With both approaches affine invariance is far from obvious and requires a separate explicit proof. Also, an important and beautiful duality (see §5.6) is obscured in both approaches. The approach taken here, using what we call the *homogeneous contour integral,* is based on the latter approach and therefore works in complete generality with minimal technical requirements. With this approach, affine or at least linear invariance is obvious at every stage, and the duality expressed by the Legendre transform can be seen easily and clearly.

One particular aspect of affine integral geometry that has been studied very little so far but deserves further attention are the affine and dual affine quermassintegrals, which were introduced by Lutwak [67, 68, 71] and proved to be affine invariant by Grinberg [39] (also, see [20, 37, 47]). Although valuations play a role in their definition, they are not valuations themselves.

This article barely scratches the surface of a deep subject of which much is known but even more is not. It is known, for example, that the affine geometric invariants described here satisfy sharp affine isoperimetric inequalities that are stronger than the classical Euclidean geometric isoperimetric inequality. These inequalities also lead to sharp affine Sobolev inequalities that are stronger than sharp Euclidean Sobolev inequalities. It is also either known or conjectured that these same invariants, when restricted to convex bodies, satisfy sharp reverse affine isoperimetric inequalities, where the extremal bodies are simplices, in contrast to the sharp affine isoperimetric inequalities, where the extremal bodies are ellipsoids. See, for example, [19, 21, 26–28, 38, 43, 44, 53, 70, 71, 73–75, 79, 85, 86, 92, 93].

I am very grateful to Erwin Lutwak, Gaoyong Zhang, and Monika Ludwig not only for their invaluable help in writing this paper but also for teaching me everything I know about affine convex geometry. I would also like to give many thanks to Franz Schuster, Christoph Haberl and the referee for a careful reading of the original draft of this paper and many improvements.

2. Basic definitions and notation

We shall suppress the use of co-ordinates and therefore the space $\mathbf{R}^n$. Instead, we will always work with an n-dimensional real vector space denoted by X.

The vector space dual to X will be denoted X^*. The natural contraction between a vector $x \in X$ and a dual vector $\xi \in X^*$ will be denoted by $\langle \xi, x \rangle = \langle x, \xi \rangle$.

We will fix a choice of Lebesgue measure m on X and denote the corresponding constant differential n-form $dm \in \Lambda^n X^*$. This, in turn, naturally induces a dual Lebesgue measure m^* and corresponding differential n-form $dm^* \in \Lambda^n X$.

Given a differentiable function $f : X \to \mathbf{R}$, recall that its differential $\partial f : X \to X^*$ is defined by

$$\langle \partial f(x), v \rangle = \left.\frac{d}{dt}\right|_{t=0} f(x + tv).$$

Since a vector space has a natural linear connection, we can also define, for a twice differentiable function f, the Hessian $\partial^2 f : X \to S^2 X^*$, where $S^2 X^*$ denotes the symmetric tensor product of X^*, by

$$\langle \partial^2 f(x), v \otimes w \rangle = \left.\frac{d}{dt}\right|_{t=0} \partial f(x + tv) w.$$

2.1. **Linear group actions.** The group of invertible linear transformations of X will be denoted by $\mathrm{GL}(X)$ or $\mathrm{GL}(n)$. If $g \in \mathrm{GL}(X)$, then the transpose of g will be denoted by $g^t : X^* \to X^*$ and its inverse by g^{-t}.

In particular, the naturally induced action of $\mathrm{GL}(X)$ on X^* is given by $\xi \mapsto g^{-t}\xi$ for each $\xi \in X^*$ and $g \in \mathrm{GL}(X)$. If Y is a subspace of a tensor product of one or more copies of X or X^*, then there is an action of $\mathrm{GL}(X)$ on Y naturally induced by the actions on X and X^*.

The naturally induced action of $\mathrm{GL}(X)$ on a function $f : X \to Y$ is $f \mapsto f_g$, where $g \in \mathrm{GL}(X)$ and $f_g(x) = gf(g^{-1}x)$ for each $x \in X$. The corresponding induced action of $\mathrm{GL}(X)$ on a function $f : X^* \to Y$ is $f \mapsto f_g$, where $f_g(\xi) = gf(g^t\xi)$ for each $\xi \in X^*$.

Each linear transformation $A : X \to X$ naturally induces a linear transformation $\det A : \Lambda^n X \to \Lambda^n X$, where $\Lambda^n X$ is the n-exterior power of X. Since $\Lambda^n X$ is 1-dimensional, $\det A \in \mathbf{R}$. Let $\mathrm{SL}(X)$ or $\mathrm{SL}(n)$ denote the group of linear transformations with $\det A = 1$. For each $d \in \mathbf{R}$, let $(\Lambda^n X)^d$ denote the d-th power of $\Lambda^n X$.

Let S^2_+X denote the space of positive definite symmetric tensor product of X and $\overline{S^2_+X}$ the space of nonnegative definite tensors. Each $A \in S^2X$ can be viewed as a linear transformation $A : X^* \to X$, and its determinant is a linear map $\det A : \Lambda^n X^* \to \Lambda^n X$ and therefore $\det A \in (\Lambda^n X)^2$.

3. Objects of study

3.1. Geometric setting. Although we will be studying geometric objects in $\mathbf{R}^n$, we want to suppress the appearance of co-ordinates.

Therefore, throughout this paper, we will denote by X an n-dimensional vector space and X^* its dual space. Note that X can be viewed as flat affine space with a distinguished point. Also, note that there is a natural isomorphism of the tangent space $T_xX \simeq X$ for each $x \in X$.

3.2. Convex body. A *body* is a compact set K in X or X^* that is the closure of its interior. We denote its boundary by ∂K. A *convex body* is a body that is convex.

For convenience we will assume in the rest of this paper that $K \subset X$ is a convex body that contains the origin in its interior.

3.3. The space of all convex bodies. A *convex body* in a vector space X is a compact convex set with nonempty interior. The set of all such bodies will be denoted $\mathcal{C}(X)$. This space is a complete metric space with respect to Hausdorff distance. It is also an Abelian semigroup under the operation of *Minkowksi sum,* which is defined by

$$K + L = \{x + y \ : \ x \in K, \ y \in L\}.$$

We will often restrict to convex bodies that contain the origin in their interiors. The set of all such bodies will be denoted $\mathcal{C}_0(X)$.

We also denote by $\mathcal{C}_0^2(X)$ the set of convex bodies in $\mathcal{C}_0(X)$ with support functions that are twice continuously differentiable (see §5.1 for the definition of a support function).

The ultimate goal is to study the affine geometric properties of convex bodies; this is equivalent to studying the affine geometric properties of compact convex hypersurfaces. Here, we will focus on geometric properties that are invariant only under affine transformations that fix a point (known as the origin). Such transformations are sometimes called *centro-affine transformations*, but we prefer the more commonly used *linear transformations.*

3.4. **Valuations.** We need an abstract framework for the study and classification of integral invariants of convex bodies. The appropriate setting is that of *valuations,* which we define here.

Given a set $\mathcal{S} \subset \mathcal{C}(X)$, a *valuation* on $\mathcal{S}$ is a finitely additive function $\Phi : \mathcal{S} \to \mathbf{R}$. In other words, if $A, B, A \cap B, A \cup B \in \mathcal{S}$, then

$$\Phi(A) + \Phi(B) = \Phi(A \cup B) + \Phi(A \cap B). \tag{1}$$

More generally, if Y is an additive semigroup, then a Y-valued valuation is a function $\Phi : \mathcal{S} \to Y$ satisfying (1).

If G is a group acting on X and $\mathcal{S} \subset \mathcal{C}(X)$ is invariant under the action of G, then a valuation $\Phi : \mathcal{S} \to \mathbf{R}$ is G-*invariant*, if $\Phi(gK) = \Phi(K)$, for each $g \in G$ and $K \in \mathcal{S}$.

If G acts on vector spaces X and Y, then a Y-valued valuation $\Phi : \mathcal{S} \to Y$ is G-*equivariant*, if $\Phi(gK) = g\Phi(K)$, for each $g \in G$ and $K \in \mathcal{S}$.

If $\mathrm{GL}(X)$ acts on Y, then a valuation $\Phi : \mathcal{S} \to Y$ is $\mathrm{GL}(X)$-*homogeneous of degree* d, if

$$\Phi(gK) = |\det g|^{d/n}\, g\Phi(K),$$

for each $g \in \mathrm{GL}(X)$ and $K \in \mathcal{S}$.

Valuations has been and still are an active area of research. For recent work that will not be discussed here, including extensions from $\mathbf{R}^n$ to more general spaces such as Riemannian manifolds, see, for example, [2–18, 22–24, 33, 34, 48–52, 80–84], as well as the references in [63].

4. Overall strategy

The standard strategy for studying a geometric object is to classify and study the properties of simpler objects that are derived from the original object and that behave appropriately under geometric transformations.

For example, differential geometers study submanifolds in Euclidean space usually by focusing on local differential invariants that are invariant under rigid motions. Invariant theory tells us that the only such invariants are the first and second fundamental forms and covariant derivatives of the second fundamental form (all covariant derivatives of the first fundamental form vanish).

We would like to follow the outline—if not the details—of such a strategy to study the affine geometry of convex bodies. In particular, our goal is to define and classify global affine invariant integral (instead of local differential) invariants of convex bodies.

The appropriate abstract setting for integral invariants turns out to be the study of *continuous valuations* on convex bodies that behave well under linear transformations, as defined above. These valuations lead naturally to affine integral invariants of a convex body. Recent classification theorems obtained by Ludwig and Reitzner show that all possible affine integral invariants are obtained by the constructions presented here.

5. Fundamental constructions

5.1. **The support function.** The *support function* of a set $K \subset X$ is the function $h_K : X^* \to \mathbf{R}$ given by

$$h_K(\xi) = \sup\{\langle \xi, x\rangle \ : \ x \in K\}.$$

Observe that h_K is convex and homogeneous of degree 1 and is strictly positive if the origin lies in the interior of the convex hull of K. Given $\xi \in X^*$, the level sets of $\xi : X \to \mathbf{R}$ comprise a family of parallel hyperplanes. The level set $\{x \ : \ \langle \xi, x\rangle = h_K(\xi)\}$ is the "last" hyperplane in the family that touches K.

If $K \in \mathcal{C}(X)$, then K can be recovered from the support function by

$$K = \{x \ : \ \langle \xi, x\rangle \leq h_K(\xi), \xi \in X^*\}.$$

The support function behaves nicely under affine transformations. In particular, given an affine transformation $x \mapsto Ax + v$ for each $x \in X$, where $A : X \to X$ is a linear transformation and $v \in X$. Then

$$h_{AK+v}(\xi) = h_K(A^t\xi) + \langle \xi, v\rangle. \tag{2}$$

The support function is also a valuation.

Lemma 5.1. *The map $K \mapsto h_K$ is a $GL(X)$-equivariant valuation from $\mathcal{C}(X)$ to the space of real-valued functions on X^*.*

Proof. If $K, L, K \cup L \in \mathcal{C}_0(X)$, then

$$\begin{aligned} h_{K\cup L} &= \max(h_K, h_L) \\ h_{K\cap L} &= \min(h_K, h_L). \end{aligned}$$

Therefore,

$$h_{K\cup L} + h_{K\cap L} = h_K + h_L.$$

$\mathrm{GL}(X)$-equivariance follows from (2). □

The same argument also implies the following for convex bodies in $\mathcal{C}_0^2(X)$.

Lemma 5.2. *For each function* $\Phi : [0, \infty) \times X \times \overline{S_+^2 X} \to Y$*, where* Y *is an additive semigroup, the map* $K \mapsto \Phi_K$*, where* $\Phi_K : X^* \to Y$ *is given by*

$$\Phi_K(\xi) = \Phi(h_K(\xi), \partial h_K(\xi), \partial^2 h_K(\xi)),$$

is a valuation from $\mathcal{C}_0^2(X)$ *to the space of* Y*-valued functions on* X^**.*

If $\mathrm{GL}(X)$ *acts on* Y *and there exists* $q \in \mathbf{R}$ *such that*

$$\Phi(t, gv, (g \otimes g)q) = (\det g)^{q/n} \Phi(t, v, q),$$

for each $g \in \mathrm{GL}(X)$ *and* $(t, v, q) \in [0, \infty) \times X \times \overline{S_+^2 X}$*, then* Φ_K *is a* $\mathrm{GL}(X)$*-homogeneous valuation.*

5.2. **The Minkowski sum.** Given sets $K, L \subset X$, we define their *Minkowski sum* to be

$$K + L = \{x + y \ : \ x \in K,\ y \in L\}.$$

If $K, L \in \mathcal{C}(X)$, then the Minkowski sum is also given by

$$h_{K+L} = h_K + h_L.$$

By this and Lemma 5.1 it follows that the identity map $K \mapsto K$ is a $\mathrm{GL}(X)$-equivariant valuation with respect to the Minkowski sum.

Firey [32] extended the notion of a Minkowski sum to that of an L_p Minkowski sum. For $p \geq 1$, the L_p *Minkowski sum* of two convex bodies is defined by

$$h^p_{K+L} = h^p_K + h^p_L.$$

By Lemma 5.2, the identity map $K \mapsto K$ is a $\mathrm{GL}(X)$-equivariant valuation from $\mathcal{C}_0(X)$ to $\mathcal{C}_0(X)$ (with respect to the L_p Minkowski sum).

5.3. **The polar body.** The *polar body* of K is the convex body $K^* \subset X^*$ given by

$$K^* = \{\xi \;:\; \langle \xi, x\rangle \leq 1 \text{ for each } x \in K\}.$$

Each $\xi \in X^*$ is in K^* if and only if K is contained in the half-space $\xi \leq 1$.

If $K \in \mathcal{C}_0(X)$, then the support functions $h_K : X \to \mathbf{R}$ and $h_{K^*} : X^* \to \mathbf{R}$ uniquely determine each other via the identities

$$h_{K^*}(x) = \sup_{\xi \in X^* \backslash \{0\}} \frac{\langle \xi, x\rangle}{h_K(\xi)} \tag{3}$$

$$h_K(\xi) = \sup_{x \in X \backslash \{0\}} \frac{\langle \xi, x\rangle}{h_{K^*}(x)}. \tag{4}$$

From this it follows that

$$\begin{aligned} K &= \{x \;:\; h_{K^*}(x) \leq 1\} \\ K^* &= \{\xi \;:\; h_K(\xi) \leq 1\}. \end{aligned} \tag{5}$$

If K is symmetric about the origin, then h_{K^*} is simply the norm on X with unit ball K and h_K is the dual norm with unit ball K^*. The support function and polar body therefore extend the concepts of a norm and its unit ball to asymmetric convex bodies.

Observe that if $K, L, K \cup L \in \mathcal{C}_0(X)$, then,

$$(K \cup L)^* = K^* \cap L^*.$$

Therefore,

Lemma 5.3. *If $\mathcal{S}^* \subset \mathcal{C}_0(X^*)$ and $\Phi : \mathcal{S}^* \to Y$ is a valuation, then so is $\Phi^* : \mathcal{S} \to Y$, where $\Phi^*(K) = \Phi(K^*)$, for each $K \in \mathcal{S}$, and*

$$\mathcal{S} = \{K \;:\; K^* \in \mathcal{S}^*\}.$$

Moreover, if Φ is $\mathrm{GL}(X^*)$-homogeneous, then Φ is $\mathrm{GL}(X)$-homogeneous.

5.4. **The inverse Gauss map.** If $K \in \mathcal{C}_0(X)$, then its support function $h = h_K$ is a convex function and therefore its differential is L^∞ and defined almost everywhere. We observe here that, where the differential of h is defined, it defines an affine invariant notion of the *inverse Gauss map* for the boundary of K.

It follows by the homogeneity of h that $\langle \xi, \partial h(\xi)\rangle = h(\xi)$ for each $\xi \in X^*$. This implies that $\partial h(\xi) \in \partial K$ for each $\xi \in X^* \backslash \{0\}$. Moreover, ξ is conormal to the boundary of K at $\partial h(\xi)$; in other words, $\xi^\perp = T_x K$, where $x = \partial h(\xi)$. For this reason we call the map $\partial h : X^* \backslash \{0\} \to \partial K$ the *inverse Gauss map.*

The precise relationship between ∂h_K and the classical Gauss map is given as follows.

Proposition 5.4. *If X has an inner product, so X^* can be identified with X, and $K \in \mathcal{C}_0(X)$ has a differentiable support function h_K, then the Gauss map $\gamma_K : \partial K \to S^{n-1}$ is given by*

$$\gamma_K(\partial h(\xi)) = \frac{\xi}{|\xi|},$$

for each $\xi \in X^* \backslash \{0\}$, where $|\xi|$ denotes the norm induced by the inner product.

5.5. **The second fundamental form.** Let $K \in \mathcal{C}_0^2(X)$ and $h = h_K$. If ∂h is the inverse Gauss map, then $\partial^2 h$ should be the inverse second fundamental form. That is indeed the case. First, observe that if $K \in \mathcal{C}_0^2(X)$, then for each $\xi \in X^* \backslash \{0\}$, $T_x K = \xi^\perp$, where $x = \partial h(\xi)$.

Since h is a convex function, $\partial^2 h$ is nonnegative definite. Moreover, since ∂h is homogeneous of degree 0, $\langle \xi, \partial^2 h(\xi) \rangle = 0$, and therefore $\partial^2 h(\xi) \in S^2 \xi^\perp$. If $\partial^2 h(\xi)\big|_{\xi^\perp}$ is positive definite, then the second fundamental form of ∂K at $x = \partial h(\xi)$ is given by

$$\mathrm{II}(x) = \left[\partial^2 h(\xi)\big|_{\xi^\perp} \right]^{-1} \in [S^2 \xi^\perp]^* = S^2(X/\mathbf{R}\xi) = S^2 T_x^* \partial K.$$

Observe that the definition of II is invariant under translations and therefore is an affine invariant. Therefore, the standard second fundamental form of a sufficiently smooth strictly convex hypersurface ∂K is well-defined in an affine setting, even though the first fundamental form and Gauss map are not.

5.6. **The Legendre transform.** The inverse Gauss map can be made more useful, if it is made homogeneous of degree 1. This is best done by introducing the following variant of the support function.

Let $H = \frac{1}{2} h_K^2$. The differential of H, $\partial H : X^* \to X$, is homogeneous of degree 1. Its Hessian, $\partial^2 H : X^* \to S^2 X$ is homogeneous of degree 0 and always nonnegative definite.

If $K \in \mathcal{C}_0^2(X)$, then the map $\partial H : X^* \backslash \{0\} \to X \backslash \{0\}$ is a C^1 diffeomorphism. We call this map the *Legendre transform.* It has the following very nice properties.

Lemma 5.5. *If $K \in \mathcal{C}_0^2(X)$, $K^* \in \mathcal{C}_0^2(X^*)$, $H = \frac{1}{2} h_K^2$, $H^* = \frac{1}{2} h_{K^*}^2$, and*

$$x = \partial H(\xi), \tag{6}$$

then the following hold:

$$h_{K^*}(x) = h_K(\xi). \tag{7}$$

$$\langle \xi, x\rangle = h_K(\xi)h_{K^*}(x) \tag{8}$$

$$\xi = \partial H^*(x) \tag{9}$$

$$\partial^2 H^*(x) = [\partial^2 H(\xi)]^{-1} \tag{10}$$

In particular, $\partial^2 H(\xi)$ and $\partial^2 H^*(x)$ are positive definite.

Proof. Denote $h = h_K$ and $h^* = h_{K^*}$.

We begin by showing that $x \in X\backslash\{0\}$ and $\xi \in X^*\backslash\{0\}$ satisfy (8) if and only if

$$x = h^*(x)\partial h(\xi). \tag{11}$$

First, equation (8) implies that the maximum is achieved in the right side of (3), and therefore ξ is a critical point of the right side of (3). This implies (11). Conversely, if x and ξ satisfy (11), then by the homogeneity of h,

$$\begin{aligned}\langle \xi, x\rangle &= \langle \xi, h^*(x)\partial h(\xi)\rangle \\ &= h^*(x)\langle \xi, \partial h(\xi)\rangle \\ &= h^*(x)h(\xi).\end{aligned}$$

Also, by (11) and the homogeneity of h^*,

$$h^*(x) = h^*(h^*(x)\partial h(\xi)) = h^*(x)h^*(\partial h(\xi)).$$

It follows that $h^*(\partial h(\xi)) = 1$. If we now set $x = \partial H(\xi) = h(\xi)\partial h(\xi)$, then

$$h^*(x) = h(h(\xi)\partial h(\xi)) = h(\xi)h(\partial h(\xi)) = h(\xi),$$

establishing (7).

Equation (9) follows by switching the roles of ξ and x in the proof above.

Equation (10) and the positive definiteness of $\partial^2 H$ and $\partial^2 H^*$ follow by differentiating each side of (9) with respect to ξ. ☐

5.7. **The curvature function.** Next, we want to define an affine invariant curvature function, roughly equivalent to Gauss curvature. The idea is to measure the nonlinearity of h using the determinant of the Hessian of the support function h_K. This, however, does not work, because the Hessian is singular. This can be seen by observing that the differential ∂h_K is homogeneous of degree 0 and therefore, by the Euler equation (14), $\langle \xi, \partial^2 h(\xi)\rangle = 0$.

This awkward situation is best circumvented by using the function H defined above. Assume $K \in \mathcal{C}^2_0(X)$. Since the differential form dm

on X is homogeneous of degree n, it follows by Lemma 6.1 (stated below) that $(\partial H)^* dm$ is an n-form homogeneous of degree n on X^*. We can therefore write

$$(\partial H)^* dm = (\det \partial^2 H) dm^*, \tag{12}$$

where the determinant is defined relative to the measures dm and dm^*. Following standard practice in convex geometry, we define the *curvature function of* K to be the function $f_K : X^* \to \mathbf{R}$ homogeneous of degree $-n-1$ such that

$$(\det \partial^2 H) = h_K^{n+1} f_K. \tag{13}$$

Since we have already shown that ∂h_K is essentially the inverse Gauss map, it should not be a surprise that the curvature function, which is obtained in spirit by taking the determinant of the Hessian of the support function, is essentially the Gauss curvature of ∂K. More precisely, we have the following.

Proposition 5.6. *If an inner product is put on* X*, giving it a Euclidean geometric structure, then the Gauss curvature* $\kappa : \partial K \to \mathbf{R}$ *of* ∂K *is given by*

$$\kappa(\partial h_K(\xi)) = \frac{|\xi|^{-n-1}}{f_K(\xi)}$$

for each $\xi \in X^* \backslash \{0\}$.

This proposition can be proved by choosing orthonormal coordinates on X such that ξ is equal to the last basis vector and representing ∂K locally as a graph over the hyperplane spanned by all but the last basis vector.

6. The homogeneous contour integral

Our description of valuations relies on the calculus of homogeneous functions and, in particular, a special type of integral, which we call the *homogeneous contour integral*, because, as for a contour integral of a meromorphic function over a closed curve in the complex plane (minus the poles), the value of the integral depends only on the homology class of the contour. The reason is also the same: the integrand is really a closed form, so the value of the integral depends only on the cohomology class represented by the form (we learned this interpretation of the homogeneous contour integral from Juan Carlos Alvarez-Paiva).

6.1. **Homogeneous functions and differential forms.** Given $t > 0$, define $D_t : X \to X$ by $D_t(x) = tx$. A function $f : X \to \mathbf{R}$ is *homogeneous of degree* e if

$$f \circ D_t = t^e f.$$

Recall that a function f is homogeneous of degree e if and only if it satisfies the Euler equation,

$$\langle df(x), x\rangle = ef(x). \tag{14}$$

A differential form or measure μ on X is *homogeneous of degree* e, if

$$D_t^*\mu = t^e\mu. \tag{15}$$

We will need the following:

Lemma 6.1. *Let X and Y be vector spaces, μ a differential form or measure homogeneous of degree e on Y, and $F : X \to Y$ a differentiable map homogeneous of degree 1. Then the differential form or measure $F^*\mu$ is homogeneous of degree e on X.*

6.2. **The homogeneous contour integral for a differential form.** By differentiating (15) with respect to t at $t = 0$, we see that if ω is an n-form homogeneous of degree 0 on X, then

$$d(x \rfloor \omega) = 0, \tag{16}$$

where $x \rfloor \omega$ denotes the interior product of the vector field x with the n-form ω, i.e. the $(n-1)$-form such that given $n-1$ vector fields $V_2, \dots, V_n$,

$$(x \rfloor \omega)(V_2, \dots, V_n) = \omega(X, V_2, \dots, V_n).$$

If an orientation is chosen for X, then we can define the *homogeneous contour integral of* ω to be

$$\oint_X \omega = \int_{\partial D} x \rfloor \omega, \tag{17}$$

where D is any bounded domain with the origin in its interior and smooth boundary ∂D. By (16), this is well-defined and independent of the domain D.

6.3. **The homogeneous contour integral for a measure.** A measure μ on X is homogeneous of degree e if for each $t > 0$, $D_t^*\mu = t^e\mu$ or, equivalently, for each compactly supported continuous function $f : X \to \mathbf{R}$,

$$\int f(tx)\, d\mu(x) = t^{-e} \int f(x)\, d\mu(x).$$

We describe below how to extend the definition (17) of a homogeneous contour integral of a homogeneous differential form of degree 0 to that of a homogeneous contour integral of a homogeneous measure of degree 0.

We call $\Omega \subset X\backslash\{0\}$ a *star domain,* if there is a positive continuous function $h : X\backslash\{0\} \to (0,\infty)$ homogeneous of degree 1 such that

$$\Omega = \{x \;:\; h(x) \leq 1\}.$$

For each $\lambda > 1$, denote

$$R_\lambda\Omega = \lambda\Omega\backslash\Omega = \{x \;:\; 1 < h(x) \leq \lambda\}.$$

Lemma 6.2. *If μ is a measure homogeneous of degree 0 on $X\backslash\{0\}$ and $\lambda > 1$, then the quantity*

$$\frac{\mu(R_\lambda\Omega)}{\log\lambda} \tag{18}$$

is independent of both $\lambda > 1$ and the star domain Ω.

Proof. Fix $\lambda > 1$. Let Ω and Ω' be star domains in X. Choose $t > 0$ so that $\lambda t\Omega' \subset \Omega$. Therefore,

$$t\Omega' \subsetneq \lambda t\Omega' \subsetneq \Omega \subsetneq \lambda\Omega.$$

Since μ is homogeneous of degree 0,

$$\begin{aligned} 0 &= \mu(\lambda\Omega\backslash\lambda t\Omega') - \mu(\Omega\backslash t\Omega') \\ &= \mu(\lambda\Omega\backslash\Omega) + \mu(\Omega\backslash\lambda t\Omega') - \mu(\Omega\backslash\lambda t\Omega') - \mu(\lambda t\Omega'\backslash t\Omega') \\ &= \mu(R_\lambda\Omega) - \mu(R_\lambda t\Omega') \\ &= \mu(R_\lambda\Omega) - \mu(R_\lambda\Omega'). \end{aligned}$$

Therefore, the quantity (18) is independent of the star domain Ω.

The lemma now follows by observing that the function $f : (0,\infty) \to (0,\infty)$ given by

$$f(t) = \mu(R_{e^t}\Omega)$$

is a monotone function satisfying the Cauchy functional equation

$$f(s+t) = f(s) + f(t).$$

and therefore (see, for example, [1]) $f(t) = ct$ for each $t \in (0,\infty)$ and a constant c. $\square$

We can therefore define the homogeneous contour integral of a measure μ homogeneous of degree 0 on X to be

$$\oint_X \mu = \frac{\mu(R_\lambda\Omega)}{\log\lambda}, \tag{19}$$

where $\lambda > 1$ and $\Omega \subset X$ is a star domain.

The theorem below gives yet another formula for the homogeneous contour integral and establishes the equivalence of the different formulas.

Theorem 6.3. *If $h : X \to [0,\infty)$ is a continuous and homogeneous of degree 1 and $\chi : (0,\infty) \to \mathbf{R}$ is a measurable function such that*

$$\int_0^\infty \chi(t)\,\frac{dt}{t} < \infty,$$

then for each measure μ homogeneous of degree 0 on X,

$$\oint_X d\mu = \left(\int_X \chi(h(x))d\mu(x)\right) \Big/ \left(\int_0^\infty \chi(t)\frac{dt}{t}\right). \tag{20}$$

In particular, the value of the right side is independent of h and χ.

If the measure μ can be written as $d\mu = f\,dm$, where the function $f : X\backslash\{0\} \to [0,\infty)$ is continuous (and homogeneous of degree $-n$) and $D \subset X$ is a compact domain with smooth boundary and the origin in its interior, then

$$\oint_X d\mu = \int_{\partial D} x \rfloor d\mu. \tag{21}$$

Remark. If we fix an inner product on X and in (21) let Ω be the Euclidean unit ball, then we obtain the formula commonly seen in convex geometry,

$$\oint_X f\,dm = \int_{S^{n-1}} f(\theta)\,d\Theta, \tag{22}$$

where S^{n-1} is the unit sphere.

Proof. We provide two different approaches to proving the theorem. If the measure ω is absolutely continuous with respect to Lebesgue measure on $X\backslash\{0\}$, then straightforward proofs can be obtained by fixing an inner product and corresponding Euclidean structure on X and using polar co-ordinates. We also provide a proof that do not rely on an inner product.

Suppose

$$d\mu = f\,dm,$$

where $f : X\backslash\{0\} \to \mathbf{R}$ is continuous and homogeneous of degree $-n$. Fix an inner product on X and define polar co-ordinates $(r,\theta) : X \to [0,\infty) \times S^{n-1}$, where $x = r\theta$, $r = |x|$, $\theta = x/|x|$, and S^{n-1} is the standard unit sphere. Rescaling the inner product if necessary, we can write $dm = r^{n-1}\,dr\,d\Theta$, where Θ is the standard volume measure on S^{n-1}.

Then

$$\begin{aligned}
\int_X \chi(h(x))f(x)\,dm(x) &= \int_0^\infty \int_{S^{n-1}} \chi(rh(\theta))r^{-n}f(\theta)r^{n-1}\,d\Theta\,dr \\
&= \int_{S^{n-1}} \left(\int_0^\infty \chi(rh(\theta))r^{-1}\,dr\right) f(\theta)\,d\Theta \\
&= \left(\int_0^\infty \chi(t)t^{-1}\,dt\right) \int_{S^{n-1}} f(\theta)\,d\Theta.
\end{aligned}$$

This proves that the right side of (20) does not depend on the function χ and by (22) is equal to (21). Equation (20) itself now follows by setting h equal to the homogeneous function of degree 1 such that $\Omega = \{h \le 1\}$ and $\chi(t)$ equal to 1 if $1 \le t \le \lambda$ and 0 otherwise.

We can, however, prove (20) in full generality without relying on a Euclidean structure. It suffices do this when χ is a piecewise constant function. Assume that there exist $t_0 < t_1 < \cdots < t_N < \infty$ and $a_1, \ldots, a_N > 0$ such that $\chi(t) = a_k$ if $t_{k-1} \le t < t_k$ for some $k \ge 1$ and 0 otherwise. Then

$$\begin{aligned}
\int \chi(h(x))d\mu(x) &= \sum_{k=1}^N a_k\mu(t_{k-1}R_{t_k/t_{k-1}}(\Omega)) \\
&= \left(\oint_X d\mu\right)\sum_{k=1}^N a_k(\log t_k - \log t_{k-1}) \\
&= \left(\oint_X d\mu\right)\int_0^\infty \chi(t)\,\frac{dt}{t}.
\end{aligned}$$

Last, we give a proof of (21) without relying on an inner product on X. Fix $\lambda > 1$ and a continuous function $h : X\backslash\{0\} \to (0,\infty)$ homogenous of degree 1. Let $\Omega = \{h \le 1\}$ be the corresponding star domain. For each $t \in (0,1)$, we define a diffeomorphism $\Phi_t : R_\lambda\Omega \to R_{\lambda/t}t\Omega$ by

$$\Phi_t(x) = \left[\frac{\lambda - h(x)}{\lambda - 1}t + \frac{h(x)-1}{\lambda - 1}\right]x.$$

By equation (19),

$$\mu(R_{\lambda/t}t\Omega) = (\log\lambda - \log t)\oint_X d\mu.$$

Differentiating both sides with respect to t at $t = 1$, it follows that

$$\left.\frac{d}{dt}\right|_{t=1} \mu(R_{\lambda/t}t\Omega) = -\oint_X d\mu. \tag{23}$$

On the other hand, if $v : X \to X$ is given by

$$v(x) = \left.\frac{d}{dt}\right|_{t=1} \Phi_t(x) \tag{24}$$
$$= \frac{\lambda - h(x)}{\lambda - 1} x$$

and L_v denotes the Lie derivative with respect to the vector field v, then by (24),

$$\left.\frac{d}{dt}\right|_{t=1} \mu(R_{\lambda/t} t\Omega) = \left.\frac{d}{dt}\right|_{t=1} \int_{R_\lambda \Omega} \Phi_t^* d\mu$$
$$= \int_{R_\lambda \Omega} L_v d\mu$$
$$= \int_{R_\lambda \Omega} d(v \rfloor d\mu)$$
$$= \int_{\lambda \partial\Omega} v \rfloor d\mu - \int_{\partial\Omega} v \rfloor d\mu$$
$$= - \int_{\partial\Omega} x \rfloor d\mu. \tag{25}$$

By (23) and (25), the homogeneous contour integral defined by (19) is equal to (21) with D equal to any star domain Ω. Since $d(x \rfloor d\mu) = 0$, it follows that equality also holds for any domain D that contains the origin in its interior. ∎

Finally, the following shows that any standard integral of a homogeneous measure over a star domain can be written as an homogeneous contour integral.

Lemma 6.4. *If $h : X \to (0, \infty)$ is continuous and homogeneous of degree 1 and μ is a measure on $X \backslash \{0\}$ homogeneous of degree $e \in (0, \infty)$, then*

$$\mu(\Omega) = \frac{1}{e} \oint_X h^{-e} d\mu,$$

where $\Omega = \{x \ : \ h(x) \leq 1\}$.

Proof. Let $\chi : [0, \infty) \to [0, \infty)$ be given by

$$\chi(t) = \begin{cases} t^e & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

for each $t \in [0, \infty)$. By (20),

$$\begin{aligned}\mu(\Omega) &= \int_\Omega d\mu \\ &= \frac{1}{e}\left(\int_X \chi(h(x))h^{-e}(x)\,d\mu(x)\right) \Big/ \left(\int_0^\infty \chi(t)t^{-1}\,dt\right) \\ &= \frac{1}{e}\oint h^{-e}d\mu.\end{aligned}$$

□

6.4. **Homogeneous integral calculus.** A remarkable fact is that all of the standard formulas of integral calculus also hold for the homogeneous contour integral. The following results can be established using their standard counterparts and (20).

The following is a homogeneous version of Stoke's theorem.

Proposition 6.5. *If η is an $(n-1)$-form homogeneous of degree 0, then $d\eta$ is an n-form homogeneous of degree 0 and*

$$\oint d\eta = 0.$$

We can therefore integrate by parts.

Corollary 6.6. *If f and g are functions on X homogeneous of degree d and $-n-d+1$ respectively, then*

$$\oint_X f(x)\partial g(x)\,dm(x) = -\oint_X g(x)\partial f(x)\,dm(x).$$

The following homogeneous version of the change of variables theorem also follows directly from the definition of an homogeneous contour integral and the standard change of variables formula.

Proposition 6.7. *If X and Y are oriented n-dimensional vector spaces, $F : X\backslash\{0\} \to Y\backslash\{0\}$ is an orientation-preserving diffeomorphism homogeneous of degree 1, and ω is an n-form homogeneous of degree 0 on Y, then $F^*\omega$ is an n-form homogeneous of degree 0 on X and*

$$\oint_X F^*\omega = \oint_Y \omega.$$

Corollary 6.8. *If X, Y, and F are as defined in Proposition 6.7, m_X and m_Y are Lebesgue measures on X and Y respectively, and $g : Y \to$ **R** is a measurable function homogeneous of degree $-n$, then*

$$\oint_Y g(y)\,dm_Y(y) = \oint_X g(F(x))\det\partial F(x)\,dm_X(x).$$

The following is a homogeneous version of the Riesz representation theorem.

Proposition 6.9. *Given a bounded linear functional ℓ on the space of continuous functions homogeneous of degree d on X, there exists a unique measure μ homogeneous of degree $-d$ on X such that*

$$\ell(f) = \oint f(x)\, d\mu(x).$$

7. An explicit construction of valuations

In this section we describe how to construct integral invariants of a convex body $K \subset X$ that behave well under $\mathrm{GL}(X)$ and show that these invariants are valuations.

We begin with the obvious.

Lemma 7.1. *If $K \mapsto \mu_K$ is a valuation from $\mathcal{C}(X)$ to Y-valued measures homogeneous of degree 0 on a vector space V, where Y is an additive semigroup, then so is*

$$K \mapsto \oint_V \mu_K.$$

The same is true, of course, for standard integrals, but we need only the homogeneous contour integral here.

By Lemma 7.1, Lemma 5.2, and Proposition 6.7,

Proposition 7.2. *Given a vector space Y and a measurable function $\phi : X^* \times (0, \infty) \times X \times \overline{S^2_+ X} \to Y$ that is homogeneous of degree $-n$, the map $\Phi : \mathcal{C}^2_0(X) \to Y$ given by*

$$\Phi(K) = \oint_{X^*} \phi(\xi, h_K(\xi), \partial h_K(\xi), \partial^2 h_K(\xi))\, dm^*(\xi), \tag{26}$$

for each $K \in \mathcal{C}^2_0(X)$, is a valuation.

If $\phi(g^t\xi, t, gv, gw) = \phi(\xi, t, v, w)$ for each $g \in \mathrm{SL}(X)$, then Φ is an $\mathrm{SL}(X)$-invariant valuation.

If there exists $d \in \mathbf{R}$ such that $\phi(g^t\xi, t, gv, gw) = (\det g)^d\, g\phi(\xi, t, v, w)$ for each $g \in \mathrm{GL}(X)$, $\xi \in X^*$, $v \in X$, and $w \in \overline{S^2_+ X}$, then Φ is a $\mathrm{GL}(X)$-homogeneous valuation.

7.1. Duality.

Proposition 7.3. *Given a function ϕ and valuation Φ as in Proposition 7.2, there is a dual function $\phi^* : X \times (0, \infty) \times X^* \times S^2_+ X^* \to Y$ homogeneous of degree $-n$ and corresponding valuation*

$$\Phi^*(K^*) = \oint_X \phi^*(x, h_{K^*}(x), \partial h_{K^*}(x), \partial^2 h_{K^*}(x))\, dm(x).$$

such that $\Phi(K) = \Phi^*(K^*)$, *for each* $K \in \mathcal{C}^2_0(X) \cap [\mathcal{C}^2_0(X^*)]^*$.

Proof. Assume that $K \in \mathcal{C}^2_0(X)$ and $K^* \in \mathcal{C}^2_0(X^*)$. The trick here is to write everything in terms of $H = \frac{1}{2}h_K^2$.

Given a function ϕ as defined in Proposition 7.2, define $\psi : X^* \times (0,\infty) \times X \times S^2_+X \to Y$ by

$$\phi(\xi, t, x, q) = \psi\left(\xi, \frac{1}{2}t^2, tx, tq + x \otimes x\right).$$

It follows by Lemma 5.5 that for each for each convex body $K \in \mathcal{C}^2_0(X)$ and $\xi \in X^*$,

$$\psi(\xi, H_K(\xi), \partial H_K(\xi), \partial^2 H_K(\xi)) = \phi(\xi, h_K(\xi), \partial h_K(\xi), \partial^2 h_K(\xi)).$$

Define $\psi^* : X \times \mathbf{R} \times X^* \times S^2_+X^* \to Y$ by

$$\psi^*(x, t, \xi, q) = \psi(\xi, t, x, q^{-1}),$$

$\phi^* : X \times \mathbf{R} \times X^* \times S^2_+X^* \to Y$ by

$$\phi^*(x, t, \xi, q) = \psi^*\left(x, \frac{1}{2}t^2, t\xi, tq + \xi \otimes \xi\right),$$

and

$$\Phi^*(K^*) = \oint_X \phi^*(x, h_{K^*}(x), \partial h_{K^*}(x), \partial^2 h_{K^*}(x))\, dm(x).$$

Then by Lemma 5.5 and Corollary 6.8, $\Phi(K) = \Phi^*(K^*)$ for each $K \in \mathcal{C}^2_0(X)$, establishing the proposition. $\square$

7.2. **Volume.** The volume of a convex body and its polar are two examples of valuations of the type constructed above. Demonstrating this is perhaps most easily done using polar coordinates, but we insist on providing a proof that does not rely on an inner product.

Let $V(\Omega)$ denote the volume of a set $\Omega \subset X$ with respect to the Lebesgue measure m. If $\Omega \subset X^*$, then $V(\Omega)$ denotes the volume with respect to the dual measure m^*.

Lemma 7.4. *If* $K \in \mathcal{C}^2_0(X)$, *then*

$$\begin{aligned} V(K) &= \frac{1}{n}\oint_X h_{K^*}^{-n}\, dm \\ &= \frac{1}{n}\oint_{X^*} h_K f_K\, dm^*. \end{aligned}$$

Proof. The first equality follows from Lemma 6.4 with $\mu = m$ and $e = n$, and the second equality follows from the first by Corollary 6.8, (12), and (13). $\square$

8. Classification of valuations

We can now use Proposition 7.2 with canonical operations on the vector space X, its dual X^*, and symmetric tensors S^2X to construct valuations that behave nicely under the action of $\mathrm{GL}(X)$ and $\mathrm{SL}(X)$. On the other hand, recent classification theorems by M. Ludwig and M. Reitzner show that this produces all possible such valuations.

9. Scalar valuations

9.1. $\mathrm{SL}(n)$**-invariant valuations.** By Proposition 7.2 we can construct $\mathrm{SL}(n)$-invariant valuations on $\mathcal{C}_0^2(X)$ as follows.

Corollary 9.1. *If $\phi : [0,\infty) \to [0,\infty)$ is continuous, then*

$$\Omega_\phi(K) = \oint_{X^*} \phi(h_K^{n+1} f_K) h_K^{-n}\, dm^*$$

defines an $\mathrm{SL}(n)$*-invariant valuation on* $\mathcal{C}_0^2(X)$.

The valuation Ω_ϕ is a generalization of affine surface area and L_p affine surface area (see Section 9.1.1).

Using the Legendre transform, we get the following dual formula for the valuation Ω_ϕ established by Ludwig [64], generalizing the theorem of Hug (see Section 9.2).

Proposition 9.2. *If $\phi : [0,\infty) \to [0,\infty)$ is continuous, $\phi^* : [0,\infty) \to [0,\infty)$ is given by*

$$\phi^*(t) = t\phi(1/t),$$

for each $t \in (0,\infty)$ and

$$\phi^*(0) = \lim_{t\to 0} t\phi(1/t),$$

then $\Omega_\phi(K) = \Omega_{\phi^*}(K^*)$ for each $K \in \mathcal{C}_0^2(X)$.

Proof. By Proposition 7.3,

$$\begin{aligned}\Omega_\phi(K) &= \oint_{X^*} \phi(h_K^{n+1} f_K) h_K^{-n}\, dm^* \\ &= \oint_X \phi(1/(h_{K^*}^{n+1} f_{K^*})) h_{K^*}^{n+1} f_{K^*} h_{K^*}^{-n}\, dm \\ &= \Omega_{\phi^*}(K^*),\end{aligned}$$

□

To extend the valuation Ω_ϕ to $\mathcal{C}_0(X)$, the function ϕ must behave reasonably at 0 and ∞. M. Ludwig and M. Reitzner [66] have established the following converse.

Theorem 9.3. *Any* $\mathrm{SL}(X)$*-invariant upper semicontinuous real-valued valuation on* $\mathcal{C}_0(X)$ *that vanishes for polytopes must be equal to* Ω_ϕ *for a concave function* ϕ *satisfying* $\lim_{t\to 0}\phi(t) = \lim_{t\to\infty}\phi(t)/t = 0$.

9.1.1. $\mathrm{GL}(X)$*-homogeneous valuations.* An obvious way to construct scalar $\mathrm{GL}(X)$-homogeneous valuations on smooth convex bodies is as follows.

Corollary 9.4. *For each* $p \in (-\infty, -n) \cup (-n, \infty]$,

$$\Omega_p(K) = \oint_{X^*} (h_K^{n+1} f_K)^{\frac{n}{n+p}} h_K^{-n}\, dm^* \tag{27}$$

defines a $\mathrm{GL}(X)$*-homogeneous valuation on* $\mathcal{C}_0^2(X)$.

Note that $\Omega_0(K) = nV(K)$ and $\Omega_\infty(K) = nV(K^*)$.

The valuation Ω_1 is known as *affine surface area* and was originally defined only for smooth convex bodies (see, for example, the monograph of Blaschke [25]). Different generalizations of affine surface area to arbitrary convex bodies were introduced by Leichtweiss [54], Schütt and Werner [96], and Lutwak [69]. The equivalence of these definitions was established by Schütt [95], Leichtweiss [55] (also, see Hug [45]), and Dolzmann-Hug [31].

Lutwak [72] introduced the more general invariant Ω_p as defined by (27), called it the L_p affine surface area, and showed that it is invariant under $\mathrm{SL}(n)$ transformations of the convex body. Lutwak also found a characterization of Ω_p that extends it to arbitrary and not just sufficiently smooth convex bodies. Hug [45] extended these results to $p > 0$, and Meyer and Werner [87] to $p > -n$.

Geometric interpretations of the L_p affine surface area of a sufficiently smooth convex body that are valid for all $p \neq -n$ have been established by Schütt and Werner [97, 98] and Werner and Ye [101].

M. Ludwig and M. Reitzner [66] have established the following converse to Corollary 9.4.

Theorem 9.5. *Any upper semicontinuous real-valued valuation* Φ *on* $\mathcal{C}_0(X)$ *that is* $\mathrm{GL}(X)$*-homogeneous of degree* $q \in \mathbf{R}$ *must be*

$$\Phi = \begin{cases} b_0\Omega_p & \text{if } q \in [-n, 0) \cup (0, n] \\ a_0 + b_0\Omega_p & \text{if } q = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $a_0 \in \mathbf{R}$ *and* $b_0 \in [0, \infty)$ *and*

$$\frac{q}{n} = \frac{n-p}{n+p}.$$

9.2. **Hug's theorem.** Hug [46] established a duality formula for the L_p affine surface area of arbitrary convex bodies. For sufficiently smooth convex bodies, Hug's theorem is easily proved using the Legendre transform.

Proposition 9.6. *If $K \in \mathcal{C}^2_0(X)$ and $K^* \in \mathcal{C}^2_0(X^*)$, then*

$$\Omega_p(K^*) = \Omega_{n^2/p}(K).$$

Proof. Set

$$\phi(t) = t^{p/(n+p)}$$
$$\phi^*(t) = t^{n/(n+p)}.$$

in Proposition 9.2. □

10. Continuous GL(n)-homogeneous valuations

In what follows, we will restrict our attention to valuations defined with the homogeneous contour integral, where the integrand is a function of a convex body's support function and its derivative or those of the polar but without using second or higher derivatives. By theorems of M. Ludwig, it turns out that this is all that is needed in order to construct all possible continuous GL(n)-homogeneous valuations of convex bodies.

10.1. **Scalar valuations.** Of the scalar valuations defined in §9, the only ones that meet the restrictions above are the volume of the convex body and the volume of its polar. There is in fact another continuous scalar valuation, namely the constant valuation ($e(K) = 1$ for each nonempty convex body K), which cannot be written as an homogeneous contour integral.

Ludwig [59] established the following:

Theorem 10.1. *A real-valued* GL(n)*-homogeneous continuous valuation on $\mathcal{C}_0(X)$ must be a constant times one of the following for each $K \in \mathcal{C}_0(X)$:*

$$K \mapsto \begin{cases} 1 \\ V(K) \\ V(K^*) \end{cases}$$

10.2. **Vector-valued valuations.** Two obvious vector-valued linear invariants of a convex body are its center of mass and the center of mass of its polar. The center of mass of $K \in \mathcal{C}(X)$ is defined to be

$$c(K) = \frac{1}{V(K)} \int_K x \, dm(x).$$

The "denormalized" center of mass $V(K)c(K)$ is a valuation that has the following formulas.

Proposition 10.2. *If $K \in \mathcal{C}_0^2(X)$, then*

$$V(K)c(K) = \int_K x\,dm(x) \tag{28}$$

$$= \frac{1}{n+1}\oint_X xh_{K^*}^{-n-1}(x)\,dm(x) \tag{29}$$

$$= \frac{1}{n+1}\oint_{X^*} \partial h_K(\xi)h_K(\xi)f_K(\xi)\,dm^*(\xi)$$

$$= \frac{n}{n+1}\int_{K^*} \partial h(\xi)h_K(\xi)^{n+1}f_K(\xi)\,dm^*(\xi)$$

Observe that (28) extends the definition of $c(K)$ to each $K \in \mathcal{C}(X)$, and (29) is valid for each $K \in \mathcal{C}_0(X)$.

In dimension 2 there is an additional twist. Observe that $A \in \Lambda^2X^*$ defines a linear map $A : X \to X^*$. Moreover, $g^tAg \in \Lambda^2X^*$ for each $g \in \mathrm{GL}(X)$. If $\dim X = 2$, then Λ^2X^* is 1-dimensional and therefore if A is nonzero, then g^tAg must be a scalar multiple of A. It is easily verified that if $\dim X = 2$, then

$$g^tAg = (\det g)A, \tag{30}$$

for each $A \in \Lambda^2X^*$ and $g \in \mathrm{GL}(X)$. This observation leads to the following:

Lemma 10.3. *Assume $\dim X = 2$ and $\mathcal{S} \subset \mathcal{C}(X)$ is invariant under $\mathrm{GL}(X)$.*

If $\Phi : \mathcal{S} \to X$ is a $\mathrm{GL}(2)$-homogeneous valuation, and $A \in \Lambda^2X^*$, then $A\Phi : \mathcal{S} \to X^*$ is a $\mathrm{GL}(2)$-homogeneous valuation.

If $\Phi : \mathcal{S} \to X^*$ is a $\mathrm{GL}(2)$-homogeneous valuation, and $A \in \Lambda^2X$, then $A\Phi : \mathcal{S} \to X$ is a $\mathrm{GL}(2)$-homogeneous valuation.

Proof. We prove only the first statement. Assume Φ is $\mathrm{GL}(2)$-homogeneous of degree $q \in \mathbf{R}$. If $g \in \mathrm{GL}(X)$, then by the definition of $\mathrm{GL}(n)$-homogeneity and (30),

$$\begin{aligned} A\Phi(gK) &= (\det g)^{q/2}Ag\Phi(K) \\ &= (\det g)^{q/2}(\det g)g^{-t}A\Phi(K) \\ &= (\det g)^{1+q/2}g^{-t}A\Phi(K). \end{aligned}$$

□

Ludwig [57] proved the following:

Theorem 10.4. *If* $\Phi : \mathcal{C}_0(X) \to X$ *is a nonvanishing* $\mathrm{GL}(n)$*-homogeneous continuous valuation, then either there exists* $a \in \mathbf{R}$ *such that* $\Phi(K) = aV(K)c(K)$ *or* $\dim X = 2$ *and there exists* $A \in \Lambda^2 X$ *such that* $\Phi(K) = V(K)Ac(K^*)$*, for each* $K \in \mathcal{C}_0(X)$*.*

If $\Phi : \mathcal{C}_0(X) \to X^*$ *is a nonvanishing* $\mathrm{GL}(n)$*-homogeneous continuous valuation, then either there exists* $a \in \mathbf{R}$ *such that* $\Phi(K) = aV(K^*)c(K^*)$ *or* $\dim X = 2$ *and there exists* $A \in \Lambda^2 X^*$ *such that* $\Phi(K) = V(K)Ac(K)$*, for each* $K \in \mathcal{C}_0(X)$*.*

There are obvious formulas for two other $\mathrm{GL}(n)$-homogeneous vector-valued valuations, but by Corollary 6.8, they are identically zero in dimensions 2 and higher. One is

$$\begin{aligned}\int_{K^*} \partial h_K(\xi)\, dm^*(\xi) &= \frac{1}{n} \oint h_K^{-n} \partial h_K \, dm^*(\xi) \\ &= -\frac{1}{n(n-1)} \oint \partial(h_K^{-n+1}) \\ &= 0.\end{aligned}$$

The other is just the same formula applied to K^*.

11. Matrix-valued valuations

A natural symmetric matrix-valued linear invariant of $K \in \mathcal{C}_0(X)$ is its second moment

$$M_2(K) = \frac{n+2}{V(K)} \int_K x \otimes x \, dm(x) \in S^2 X.$$

The normalization constant $n+2$ is chosen so that if K is the unit ball of the inner product $g \in S^2 X^*$, then $M_2(K) = g^{-1} \in S^2$.

A new symmetric matrix-valued linear invariant of $K \in \mathcal{C}_0(X)$ was introduced in [74] (and later identified as being analogous to the Fisher information of a probability density). This is defined to be

$$M_{-2}(K) = \frac{n}{V(K)} \int_K \partial h_{K^*}(x) \otimes \partial h_{K^*}(x)\, dm(x) \in S^2 X^*.$$

Note that $M_{-2}(K)$ is *not* necessarily equal to $M_2(K^*)$. Again, the normalization constant n is chosen so that if K is the unit ball of the inner product $g \in S^2 X^*$, then $M_{-2}(K) = g$.

The denormalized invariants are valuations given by the following formulas.

Proposition 11.1. *If $K \in C_0^2(X)$, then*

$$V(K)M_2(K) = (n+2)\int_K x \otimes x \, dm(x) \tag{31}$$

$$= \oint_X (x \otimes x) h_{K^*}^{-n-2}(x)\, dm(x) \tag{32}$$

$$= \oint_{X^*} [\partial h_{K^*}(\xi) \otimes \partial h_{K^*}(\xi)] h_K(\xi) f_K(\xi)\, dm^*(\xi)$$

$$= n\int_{K^*} [\partial h_{K^*}(\xi) \otimes \partial h_{K^*}(\xi)] h_K^{n+1}(\xi) f_K(\xi)\, dm^*(\xi).$$

and

$$V(K)M_{-2}(K) = n\int_K \partial h_{K^*}(x) \otimes \partial h_{K^*}(x)\, dm(x) \tag{33}$$

$$= \oint_X [\partial h_{K^*}(x) \otimes \partial h_{K^*}(x)] h_{K^*}^{-n}(x)\, dm(x) \tag{34}$$

$$= \oint_{X^*} (\xi \otimes \xi) h_K^{-1}(\xi) f_K(\xi)\, dm(x)$$

$$= (n+2)\int_{K^*} (\xi \otimes \xi) h_K^{n+1}(\xi) f_K(\xi)\, dm^*(\xi).$$

Equations (31) and (33) extend the definition of $M_2(K)$ and $M_{-2}(K)$ to each $K \in \mathcal{C}(X)$, while (32) and (34) are valid for each $K \in \mathcal{C}_0(X)$.

In [60] Ludwig classifies symmetric matrix-valued valuations. Ludwig [60] established the following:

Theorem 11.2. *If $\Phi : \mathcal{C}_0(X) \to S^2X$ is a nonvanishing* $\mathrm{GL}(n)$*-homogeneous continuous valuation, then there exists $a \in \mathbf{R}$ such that either $\Phi(K) = aV(K)M_2(K)$ or $\Phi(K) = aV(K^*)M_{-2}(K^*)$.*

If $\Phi : \mathcal{C}_0(X) \to S^2X^*$ is a nonvanishing $\mathrm{GL}(n)$*-homogeneous continuous valuation, then there exists $a \in \mathbf{R}$ such that either $\Phi(K) = aV(K)M_{-2}(K)$ or $\Phi(K) = aV(K^*)M_2(K^*)$.*

11.1. **The Cramer-Rao inequality.** The Cramer-Rao inequality [30, 91] states that the second moment matrix of a random vector dominates the inverse of the Fisher information matrix, and equality holds if and only if the random vector is Gaussian. The proof (see, for example, [29]) is elementary, requiring only the Cauchy-Schwarz inequality and integration by parts.

In [75] a Cramer-Rao inequality is proved for convex bodies. The proof given in [75] appears to be much more involved than the one for random vectors. We show here that it suffices to give essentially the same proof as for random vectors but with all integrals replaced by homogeneous contour integrals.

First, we recall the original Cramer-Rao inequality and its proof.

Theorem 11.3. *[30, 91] If R is a random vector in X with probability measure $f\,dm$, where $f : X \to [0,\infty)$ is differentiable and decays sufficiently rapidly at infinity, then*

$$E[R \otimes R] \geq E[\partial \log f(R) \otimes \partial \log f(R)]^{-1}, \tag{35}$$

with equality holding if and only if R is Gaussian.

Proof. If $v \in X$ and $\xi \in X^*$, then by integration by parts and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\langle \xi, v \rangle &= \int_X [\langle v, \partial \rangle, \langle \xi, x \rangle] f(x)\, dm(x) \\
&= -\int_X \langle \xi, x \rangle \langle v, \partial f(x) \rangle\, dm(x) \\
&= -\int_X \langle \xi, x \rangle \langle v, \partial \log f(x) \rangle f(x)\, dm(x) \\
&\leq \left(\int_X \langle \xi, x \rangle^2 f(x)\, dm(x) \right)^{1/2} \left(\int_X \langle v, \partial \log f(x) \rangle^2 f(x)\, dm(x) \right)^{1/2} \\
&= (\langle \xi \otimes \xi, E[R \otimes R] \rangle \langle v \otimes v, E[\partial \log f(R) \otimes \partial \log f(R)] \rangle)^{1/2}.
\end{aligned}$$

Since this holds for all $v \in X$ and $\xi \in X^*$, inequality (35) follows. □

Next, we adapt the proof above to convex bodies.

Theorem 11.4. *[75] If $K \in \mathcal{C}_0(X)$, then*

$$M_2(K) \geq M_{-2}(K)^{-1}, \tag{36}$$

with equality holding if and only if K is an ellipsoid centered at the origin.

Proof. If $K \in \mathcal{C}_0(X)$, $v \in X$ and $\xi \in X^*$, then by Corollary 6.6 and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\langle \xi, v \rangle &= \frac{1}{nV(K)} \oint_X [\langle v, \partial \rangle, \langle \xi, x \rangle] h_{K^*}^{-n}(x)\, dm(x) \\
&= -\frac{1}{V(K)} \oint_X \langle \xi, x \rangle \langle v, \partial h_{K^*}(x) \rangle h_{K^*}^{-n-1}(x)\, dm(x) \\
&\leq \left(\frac{1}{V(K)} \oint_X \langle \xi, x \rangle^2 h_{K^*}^{-n-2}(x)\, dm(x) \right)^{1/2} \left(\frac{1}{V(K)} \oint_X \langle v, \partial h_{K^*}(x) \rangle^2 h_{K^*}^{-n}(x)\, dm(x) \right)^{1/2} \\
&= \langle \xi \otimes \xi, M_2(K) \rangle^{1/2} \langle v \otimes v, M_{-2}(K) \rangle^{1/2}.
\end{aligned}$$

Since this holds for each $v \in X$ and $\xi \in X^*$, inequality (36) follows. $\square$

12. Homogeneous function- and convex body-valued valuations

Observe that associated with $M_2(K)$ and $M_{-2}(K)$ are homogeneous functions $h_{\Gamma_2 K} : X^* \to [0, \infty)$ given by

$$h^2_{\Gamma_2 K}(\xi) = \langle \xi \otimes \xi, M_2(K) \rangle$$

and $h_{\Gamma^*_{-2} K} : X \to [0, \infty)$ given by

$$h^2_{\Gamma^*_{-2} K}(v) = \langle v \otimes v, M_{-2}(K) \rangle.$$

The functions $h_{\Gamma_2 K}$ and $h_{\Gamma^*_{-2} K}$ are in turn the support functions of ellipsoids, which we denote by $\Gamma_2 K \subset X$ and $\Gamma^*_{-2} K \subset X^*$. These functions and bodies have the following natural generalizations.

If $p \in [1, \infty)$, let $\mathcal{H}^p_c(X)$ denote the space of convex real-valued functions on X homogeneous of degree p. Note that a function $h : X^* \to \mathbf{R}$ is the support function of a convex body that contains the origin in its interior if and only if $h^p \in \mathcal{H}^p_c(X^*)$ and is positive outside the origin.

If $K \in \mathcal{C}_0(X)$, then a natural way to define convex homogeneous functions $h^p_{\Gamma_p K} \in H^p_c(X^*)$ and $h^p_{\Gamma^*_{-p} K} \in H^p_c(X^*)$ is by

$$\begin{aligned} h^p_{\Gamma_p K}(\xi) &= \frac{c_{n,p}}{V(K)} \int_K |\langle \xi, x \rangle|^p \, dm(x) \\ h^p_{\Gamma^*_{-p} K}(v) &= \frac{c_{n,p}}{V(K)} \int_K |\langle v, \partial H_{K^*}(x) \rangle|^p \, dm(x), \end{aligned}$$

where

$$c_{n,p} = \frac{\omega_2 \omega_n \omega_{p-1}}{\omega_{n+p}},$$

and

$$\omega_k = \frac{\pi^{k/2}}{\Gamma(1 + \frac{k}{2})}$$

is the volume of the standard unit ball in k-dimensional Euclidean space. The normalization is again chosen so that if K is the unit ball of an inner product, then $\Gamma_p K = \Gamma_{-p} K = K$.

Such functions are always even and can be generalized as follows. Given $p \in [1, \infty)$, $t \in [-1, 1]$, and $K \in \mathcal{C}^2_0(X)$, define convex bodies

$\Gamma^t_p K, \Gamma^t_{-p} K \subset X$ such that

$$\begin{aligned}
h^p_{\Gamma^t_p K}(\xi) &= \frac{c_{n,p}}{V(K)} \int_K [(1+t)\langle \xi, x\rangle_+^p + (1-t)\langle \xi, x\rangle_-^p]\, dm(x) \\
&= \frac{c_{n,p}}{(n+p)V(K)} \oint_X [(1+t)\langle \xi, x\rangle_+^p + (1-t)\langle \xi, x\rangle_-^p] h_{K^*}^{-n-p}(x)\, dm(x) \\
&= \frac{c_{n,p}}{(n+p)V(K)} \oint_{X^*} [(1+t)\langle \xi, \partial H_K(\eta)\rangle_+^p + (1-t)\langle \xi, \partial H_K(\eta)\rangle_-^p] h_K(\eta) f_K(\eta)\, dm^*(\eta) \\
&= \frac{c_{n,p}}{V(K)} \oint_{K^*} [(1+t)\langle \xi, \partial H_K(\eta)\rangle_+^p + (1-t)\langle \xi, \partial H_K(\eta)\rangle_-^p] dm^*(\eta) \\
h^p_{\Gamma^{t*}_{-p} K}(v) &= \frac{c_{n,p}}{V(K)} \int_K [(1+t)\langle v, \partial H_{K^*}(x)\rangle_+^p + (1-t)\langle v, \partial H_{K^*}(x)\rangle_-^p]\, dm(x) \\
&= \frac{c_{n,p}}{(n+p)V(K)} \oint_X [(1+t)\langle v, \partial H_{K^*}(x)\rangle_+^p + (1-t)\langle v, \partial H_{K^*}(x)\rangle_-^p] h_{K^*}^{-n-p}(x)\, dm(x) \\
&= \frac{c_{n,p}}{(n+p)V(K)} \oint_{X^*} [(1+t)\langle v, \xi\rangle_+^p + (1-t)\langle v, \xi\rangle_-^p] h_K^{1-p}(\xi) f_K(\xi)\, dm^*(\xi) \\
&= \frac{c_{n,p}}{V(K)} \oint_{K^*} [(1+t)\langle v, \xi\rangle_+^p + (1-t)\langle v, \xi\rangle_-^p] h_K^{n+1}(\xi) f_K(\xi)\, dm^*(\xi)
\end{aligned}$$

where for each $s \in \mathbf{R}$, $s_+ = s$ and $s_- = 0$ if $s \geq 0$ and $s_+ = 0$ and $s_- = -s$ if $s \leq 0$.

Recent classification theorems of Ludwig [61], similar to the other theorems cited in this paper, suggest strongly the following conjecture of Ludwig: If $n \geq 3$, then any continuous $\mathrm{GL}(n)$-homogeneous $\mathcal{H}^p_c(X^*)$-valued valuation on $\mathcal{C}_0(X)$ is of the form

$$aV(K) h^p_{\Gamma^t_p K} \text{ or } aV(K^*) h^p_{\Gamma^t_{-p} K^*}$$

for some $t \in [-1, 1]$ and $a \in [0, \infty)$ and that any continuous $\mathrm{GL}(n)$-homogeneous $\mathcal{H}^p_c(X)$-valued valuation on $\mathcal{C}_0(X)$ is of the form

$$aV(K) h^p_{\Gamma^t_{-p} K} \text{ or } aV(K^*) h^p_{\Gamma^t_p K^*}$$

for some $t \in [-1, 1]$ and $a \in [0, \infty)$.

13. Questions

We end with a few rather open-ended questions.

- The discussion here exploits Proposition 6.7 and Corollary 6.8, where the change of variables is always either a linear transformation of X or the Legendre transform associated with a convex body. Are there further applications of these results, where more general diffeomorphisms homogeneous of degree 1 are used?

- Symmetric 2-tensor-valued valuations have been studied extensively. Do continuous $\mathrm{GL}(n)$-homogeneous anti-symmetric 2-tensor-valued valuations exist? (Probably not)
- Another obvious way to define a continuous $\mathrm{GL}(n)$-homogeneous matrix-valued valuation $\Phi : \mathcal{C}^2_0(X) \to X \otimes X^*$ is

$$\Phi(K) = \frac{1}{n} \oint_X x \otimes \partial h_{K^*}(x) h_{K^*}^{-n-1}(x)\, dm(x),$$

for each convex body K. The trace of this valuation is the volume of K. What if anything can be said about this valuation? Is there a classification of continuous $\mathrm{GL}(n)$-homogeneous $X \otimes X^*$-valued valuations? The conjecture would be that the only possibilities are $K \mapsto \Phi(K)$ or $\Phi(K^*)$.
- The theorems of Ludwig on the classification of continuous $\mathrm{GL}(n)$-homogeneous valuations are proved by analyzing the valuations when restricted to polytopes. Is there a different proof using the properties of the homogeneous contour integral?
- The approach taken here leads naturally to the definition of local geometric invariants of a Finsler manifold, by using the homogeneous contour integral on each tangent or cotangent space. For example, if $K_x \subset T_x M$ is the unit ball of a Finsler metric at a point x in the manifold M, then $\Gamma_2 K_x$, $\Gamma_{-2} K_x$, $\Gamma_2 K_x^*$, and $\Gamma_{-2} K_x^*$ define different Riemannian metrics naturally associated with the Finsler metric. Are such invariants useful in Finsler geometry?
- It has been shown [76–78] that affine valuations of convex bodies are naturally associated with moments and generalized Fisher information (also known as Sobolev norms) of probability density functions. Can the concept of valuations be extended to probability densities or measures and classified in that context?

References

1. J. Aczél, *Lectures on functional equations and their applications*, Mathematics in Science and Engineering, Vol. 19, Academic Press, New York, 1966.
2. S. Alesker, *Continuous valuations on convex sets*, Geom. Funct. Anal. **8** (1998), no. 2, 402–409. MR 1616167 (99b:52027)
3. ———, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. (2) **149** (1999), no. 3, 977–1005. MR 1709308 (2000i:52019)
4. ———, *On P. McMullen's conjecture on translation invariant valuations*, Adv. Math. **155** (2000), no. 2, 239–263. MR 1794712 (2001k:52013)

5. ______, *Classification results on valuations on convex sets*, European Congress of Mathematics, Vol. II (Barcelona, 2000), Progr. Math., vol. 202, Birkhäuser, Basel, 2001, pp. 1–8. MR 1905346 (2003e:52023)
6. ______, *Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture*, Geom. Funct. Anal. **11** (2001), no. 2, 244–272. MR 1837364 (2002e:52015)
7. ______, *Algebraic structures on valuations, their properties and applications*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 757–764. MR 1957082 (2004b:52022)
8. ______, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom. **63** (2003), no. 1, 63–95. MR 2015260 (2004h:52015)
9. ______, *Hard Lefschetz theorem for valuations and related questions of integral geometry*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1850, Springer, Berlin, 2004, pp. 9–20. MR 2087146 (2005h:52012)
10. ______, *The multiplicative structure on continuous polynomial valuations*, Geom. Funct. Anal. **14** (2004), no. 1, 1–26. MR 2053598 (2005d:52022)
11. ______, SU(2)-*invariant valuations*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1850, Springer, Berlin, 2004, pp. 21–29. MR 2087147 (2005f:52029)
12. ______, *Valuations on convex sets, non-commutative determinants, and pluripotential theory*, Adv. Math. **195** (2005), no. 2, 561–595. MR 2146354 (2006f:32047)
13. ______, *Theory of valuations on manifolds. I. Linear spaces*, Israel J. Math. **156** (2006), 311–339. MR 2282381 (2008b:52018)
14. ______, *Theory of valuations on manifolds. II*, Adv. Math. **207** (2006), no. 1, 420–454. MR 2264077 (2008b:52019)
15. ______, *Erratum: "Continuous rotation invariant valuations on convex sets" [Ann. of Math. (2)* **149** *(1999), no. 3, 977–1005]*, Ann. of Math. (2) **166** (2007), no. 3, 947–948. MR 2373377
16. ______, *Theory of valuations on manifolds: a survey*, Geom. Funct. Anal. **17** (2007), no. 4, 1321–1341. MR 2373020
17. ______, *Theory of valuations on manifolds. IV. New properties of the multiplicative structure*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1910, Springer, Berlin, 2007, pp. 1–44. MR 2347038
18. S. Alesker and J. H. G. Fu, *Theory of valuations on manifolds. III. Multiplicative structure in the general case*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 1951–1981. MR 2366970
19. K. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. (2) **44** (1991), 351–359.
20. J. Bastero, J. Bernués, and M. Romance, *Dual quermassintegrals, extremal positions and isotropic measures*, Seminar of Mathematical Analysis, Univ. Sevilla Secr. Publ., Seville, 2006, pp. 11–26. MR 2276958 (2008g:52013)
21. J. Bastero and M. Romance, *Positions of convex bodies associated to extremal problems and isotropic measures*, Adv. Math. **184** (2004), no. 1, 64–88.

22. A. Bernig, *Support functions, projections and Minkowski addition of Legendrian cycles*, Indiana Univ. Math. J. **55** (2006), no. 2, 443–464. MR MR2225441 (2007i:53083)
23. ———, *Valuations with Crofton formula and Finsler geometry*, Adv. Math. **210** (2007), 733–753. MR MR2303237 (2008i:53108)
24. A. Bernig and L. Bröcker, *Valuations on manifolds and Rumin cohomology*, J. Differential Geom. **75** (2007), 433–457. MR MR2301452 (2008e:53146)
25. W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. II. Affine Differentialgeometrie, bearbeitet von K. Reidemeister. Erste und zweite Auflage.*, J. Springer, Berlin, 1923.
26. J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbf{R}^n$, Invent. Math. **88** (1987), no. 2, 319–340.
27. S. Campi and P. Gronchi, *The* L^p*-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128–141.
28. ———, *On the reverse* L^p*-Busemann-Petty centroid inequality*, Mathematika **49** (2002), 1–11 (2004).
29. T. M. Cover and J. A. Thomas, *Elements of information theory*, John Wiley & Sons Inc., New York, 1991, A Wiley-Interscience Publication.
30. H. Cramér, *Mathematical methods of statistics*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, Reprint of the 1946 original. MR 2001k:62002
31. G. Dolzmann and D. Hug, *Equality of two representations of extended affine surface area*, Arch. Math. (Basel) **65** (1995), 352–356. MR 1349190 (97c:52019)
32. Wm. J. Firey, *p-means of convex bodies*, Math. Scand. **10** (1962), 17–24.
33. J. H. G. Fu, *Integral geometry Alesker's theory of valuations*, Integral geometry and convexity, World Sci. Publ., Hackensack, NJ, 2006, pp. 17–27. MR 2240970 (2007h:52016)
34. ———, *Structure of the unitary valuation algebra*, J. Differential Geom. **72** (2006), no. 3, 509–533. MR 2219942 (2007b:52008)
35. R. J. Gardner, *Geometric Tomography*, Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 1995.
36. ———, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), 355–405.
37. ———, *The dual Brunn-Minkowski theory for bounded Borel sets: dual affine quermassintegrals and inequalities*, Adv. Math. **216** (2007), 358–386. MR 2353261
38. Y. Gordon, M. Meyer, and S. Reisner, *Zonoids with minimal volume-product—a new proof*, Proc. Amer. Math. Soc. **104** (1988), no. 1, 273–276.
39. E. L. Grinberg, *Isoperimetric inequalities and identities for* k*-dimensional cross-sections of convex bodies*, Math. Ann. **291** (1991), no. 1, 75–86. MR 1125008 (92g:52008)
40. C. Haberl, *Blaschke valuations*, preprint, 2009.
41. ———, *Star body valued valuations*, Indiana Univ. Math. J. (2009), to appear.
42. C. Haberl and M. Ludwig, *A characterization of* L_p *intersection bodies*, Int. Math. Res. Not. (2006), Art. ID 10548, 29.

43. C. Haberl and F. Schuster, *General L^p affine isoperimetric inequalities*, J. Diff. Geom. (2008), to appear (arXiv:0809.1983).
44. ______, *Asymmetric affine L^p Sobolev inequalities*, J. Funct. Anal. (2009), to appear.
45. D. Hug, *Contributions to affine surface area*, Manuscripta Math. **91** (1996), 283–301. MR 1416712 (98d:52009)
46. ______, *Curvature relations and affine surface area for a general convex body and its polar*, Results Math. **29** (1996), no. 3-4, 233–248. MR 1387565 (97c:52004)
47. Y. Jun and L. Gangsong, *Inequalities for dual affine quermassintegrals*, J. Inequal. Appl. (2006), Art. ID 50181, 7. MR 2221229 (2007f:52024)
48. D. A. Klain, *A short proof of Hadwiger's characterization theorem*, Mathematika **42** (1995), no. 2, 329–339. MR 1376731 (97e:52008)
49. ______, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), no. 1, 80–101. MR 1399604 (97i:52009)
50. ______, *Invariant valuations on star-shaped sets*, Adv. Math. **125** (1997), no. 1, 95–113. MR 1427802 (98a:52009)
51. ______, *Even valuations on convex bodies*, Trans. Amer. Math. Soc. **352** (2000), no. 1, 71–93. MR 1487620 (2000c:52003)
52. ______, *Isometry-invariant valuations on hyperbolic space*, Discrete Comput. Geom. **36** (2006), no. 3, 457–477. MR 2255514 (2007g:52006)
53. G. Kuperberg, *From the Mahler conjecture to Gauss linking integrals*, Geom. Funct. Anal. **18** (2008), 870–892.
54. K. Leichtweiß, *Zur Affinoberfläche konvexer Körper*, Manuscripta Math. **56** (1986), 429–464. MR 860732 (87k:52011)
55. ______, *Bemerkungen zur Definition einer erweiterten Affinoberfläche von E. Lutwak*, Manuscripta Math. **65** (1989), 181–197. MR 1011431 (90j:52004)
56. A. M. Li, U. Simon, and G. S. Zhao, *Global affine differential geometry of hypersurfaces*, de Gruyter Expositions in Mathematics, vol. 11, Walter de Gruyter & Co., Berlin, 1993.
57. M. Ludwig, *Moment vectors of polytopes*, Rend. Circ. Mat. Palermo (2) Suppl. (2002), no. 70, part II, 123–138, IV International Conference in "Stochastic Geometry, Convex Bodies, Empirical Measures & Applications to Engineering Science", Vol. II (Tropea, 2001).
58. ______, *Projection bodies and valuations*, Adv. Math. **172** (2002), 158–168.
59. ______, *Valuations of polytopes containing the origin in their interiors*, Adv. Math. **170** (2002), 239–256.
60. ______, *Ellipsoids and matrix-valued valuations*, Duke Math. J. **119** (2003), 159–188.
61. ______, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213.
62. ______, *Intersection bodies and valuations*, Amer. J. Math. **128** (2006), no. 6, 1409–1428.
63. ______, *Valuations in the affine geometry of convex bodies*, Integral geometry and convexity, World Sci. Publ., Hackensack, NJ, 2006, pp. 49–65. MR 2240973 (2007f:52033)
64. ______, *General affine surface areas*, preprint, 2008.

65. M. Ludwig and M. Reitzner, *A characterization of affine surface area*, Adv. Math. **147** (1999), no. 1, 138–172.
66. ______, *A classification of SL(n) invariant valuations*, Annals Math. (2009), to appear.
67. E. Lutwak, *A general isepiphanic inequality*, Proc. Amer. Math. Soc. **90** (1984), no. 3, 415–421. MR 728360 (85i:52005)
68. ______, *Inequalities for Hadwiger's harmonic Quermassintegrals*, Math. Ann. **280** (1988), no. 1, 165–175. MR 928304 (89h:52005)
69. ______, *Extended affine surface area*, Adv. Math. **85** (1991), 39–68. MR 1087796 (92d:52012)
70. ______, *Inequalities for mixed projection bodies*, Trans. Amer. Math. Soc. **339** (1993), 901–916. MR MR1124171 (93m:52011)
71. ______, *Selected affine isoperimetric inequalities*, Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 151–176. MR 1242979 (94h:52014)
72. ______, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294. MR MR1378681 (97f:52014)
73. E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.
74. ______, *A new ellipsoid associated with convex bodies*, Duke Math. J. **104** (2000), 375–390.
75. ______, *The Cramer-Rao inequality for star bodies*, Duke Math. J. **112** (2002), 59–81.
76. ______, *Cramer-Rao and moment-entropy inequalities for Renyi entropy and generalized Fisher information*, IEEE Trans. Inform. Theory **51** (2005), 473–478.
77. ______, *Optimal Sobolev norms and the L^p Minkowski problem*, Int. Math. Res. Not. (2006), 62987, 1–21.
78. ______, *Moment-entropy inequalities for a random vector*, IEEE Trans. Inform. Theory **53** (2007), 1603–1607. MR MR2303029 (2007m:94071)
79. E. Lutwak and G. Zhang, *Blaschke-Santalo inequalities*, J. Differential Geom. **47** (1997), 1–16.
80. P. McMullen, *Continuous translation-invariant valuations on the space of compact convex sets*, Arch. Math. (Basel) **34** (1980), 377–384. MR MR593954 (81m:52013)
81. ______, *Valuations and dissections*, Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 933–988. MR MR1243000 (95f:52018)
82. ______, *Isometry covariant valuations on convex bodies*, Rend. Circ. Mat. Palermo (2) Suppl. (1997), 259–271, II International Conference in "Stochastic Geometry, Convex Bodies and Empirical Measures" (Agrigento, 1996). MR MR1602986 (99a:52016)
83. ______, *Valuations and tensor weights on polytopes*, Mathematika **53** (2006), 1–47 (2007). MR MR2304051 (2008d:52018)
84. ______, *Valuations on lattice polytopes*, Adv. Math. **220** (2009), 303–323. MR MR2462842
85. M. Meyer, *Convex bodies with minimal volume product in $\mathbf{R}^2$*, Monatsh. Math. **112** (1991), no. 4, 297–301.

86. M. Meyer and S. Reisner, *Shadow systems and volumes of polar convex bodies*, Mathematika **53** (2006), no. 1, 129–148 (2007).
87. M. Meyer and E. Werner, *On the p-affine surface area*, Adv. Math. **152** (2000), 288–313. MR MR1764106 (2001g:52012)
88. K. Nomizu, *Recent results in affine differential geometry—an introduction and a survey*, Geometry and global analysis (Sendai, 1993), Tohoku Univ., Sendai, 1993, pp. 351–357. MR 1361200 (96k:53012)
89. ______, *Recent developments in affine differential geometry*, Sūgaku **46** (1994), no. 4, 308–322. MR 1336128 (96b:53019)
90. K. Nomizu and T. Sasaki, *Affine differential geometry*, Cambridge Tracts in Mathematics, vol. 111, Cambridge University Press, Cambridge, 1994, Geometry of affine immersions. MR 1311248 (96e:53014)
91. C. Radhakrishna Rao, *Information and the accuracy attainable in the estimation of statistical parameters*, Bull. Calcutta Math. Soc. **37** (1945), 81–91. MR 7,464a
92. S. Reisner, *Zonoids with minimal volume-product*, Math. Z. **192** (1986), no. 3, 339–346.
93. J. Saint-Raymond, *Sur le volume des corps convexes symétriques*, Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981, Publ. Math. Univ. Pierre et Marie Curie, vol. 46, Univ. Paris VI, Paris, 1981, pp. Exp. No. 11, 25.
94. R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.
95. C. Schütt, *On the affine surface area*, Proc. Amer. Math. Soc. **118** (1993), 1213–1218. MR 1181173 (93j:52009)
96. C. Schütt and E. Werner, *The convex floating body*, Math. Scand. **66** (1990), 275–290. MR 1075144 (91i:52005)
97. ______, *Polytopes with vertices chosen randomly from the boundary of a convex body*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1807, Springer, Berlin, 2003, pp. 241–422.
98. C. Schütt and E. Werner, *Surface bodies and p-affine surface area*, Adv. Math. **187** (2004), 98–145.
99. U. Simon, *Affine differential geometry*, Handbook of differential geometry, Vol. I, North-Holland, Amsterdam, 2000, pp. 905–961.
100. A. C. Thompson, *Minkowski geometry*, Encyclopedia of Mathematics and its Applications, vol. 63, Cambridge University Press, Cambridge, 1996.
101. E. Werner and D. Ye, *Inequalities for mixed p-affine surface area*, arXiv:0812.4550v1 [math.MG], 2008.

Department of Mathematics, Polytechnic Institute of NYU, Brooklyn, New York 11201