AFFINE INTEGRAL GEOMETRY FROM A DIFFERENTIABLE VIEWPOINT

DEANE YANG

I dedicate this to the sixtieth birthday of Shing Tung Yau

Abstract. The notion of a homogeneous contour integral is introduced and used to construct affine integral invariants of convex bodies. Earlier descriptions of this construction rely on a Euclidean structure on the ambient vector space, so the behavior under linear transformations is established after the fact. The description presented here relies only on the linear structure and a Lebesgue measure on the ambient vector space, making its behavior under linear transformations more transparent.

Also, recent work by Ludwig classifying different types of affine or linearly invariant valuations on convex bodies is reviewed. The invariants obtained using the homogeneous contour integral are exactly the invariants that arise in Ludwig’s classification theorems.

1. Introduction

The subject of affine geometry can be described as the study of affine invariant properties of subsets of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Here, affine invariant means invariant under the group of affine transformations, which is the group generated by linear transformations and translations.

There are two distinct approaches to this subject. One is known as affine differential geometry. This approach is analogous to the one taken in Euclidean or Riemannian differential geometry. One studies affine invariant local differential invariants of the set or its boundary. Recent accounts of affine differential geometry include [56, 88–90, 99]. This approach will not be discussed here.

Instead, we will describe here one aspect of affine integral geometry, where one studies affine geometric invariants of a convex body in \( \mathbb{R}^n \) obtained by integration or averaging. Although this is an active

Date: February 2, 2009, revised March 29, 2009 and April 6, 2009.

2000 Mathematics Subject Classification. 52A22, 53C65, 52B45, 52A39, 52A20.

Key words and phrases. convex body, affine invariant, valuation, integral geometry, convex geometry, affine geometry.
area of research, it is perhaps not familiar to most differential geom-
ters. There is no shortage of surveys and research monographs on the
subject, including books by Schneider [94], Gardner [35], and Thomp-
son [100], as well as an excellent survey article by Gardner [36].

In this short article, we will focus on recent progress [57–63,65] to-
wards the classification of all possible affine integral invariants of a
convex body. These results will be presented from a viewpoint that
might be more comfortable and easier to digest for differential geom-
ters but that is not seen in most writings on affine integral geometry.
Ludwig [63] has written another and particularly beautiful survey tak-
ing a different approach to the same topics. Ludwig-Haberl [42] and
Haberl [40, 41] have also established classification theorems with as-
sumptions different from those considered here. Also worth noting is
McMullen’s classification of translation invariant valuations [81].

Another motivation for this article is to present a new approach,
more transparent than other commonly used ones, to constructing
affine integral invariants of a convex body. There are two previously
known approaches. One, favored by differential geometers and analysts,
is to construct appropriate measures, usually involving a generalized
notion of curvature, on the boundary of the body and integrate over
them. There are, however, significant technical difficulties in work-
ing with such measures for a general convex body. The second less
technical approach, favored by convex geometric analysts, is to con-
struct appropriate measures on the unit sphere (viewed as the space
of all possible outward unit normals to the boundary). With both ap-
proaches affine invariance is far from obvious and requires a separate
explicit proof. Also, an important and beautiful duality (see §5.6) is
obscured in both approaches. The approach taken here, using what we
call the homogeneous contour integral, is based on the latter approach
and therefore works in complete generality with minimal technical re-
quirements. With this approach, affine or at least linear invariance
is obvious at every stage, and the duality expressed by the Legendre
transform can be seen easily and clearly.

One particular aspect of affine integral geometry that has been stud-
ied very little so far but deserves further attention are the affine and
dual affine quermassintegrals, which were introduced by Lutwak [67,
68, 71] and proved to be affine invariant by Grinberg [39] (also, see
[20, 37, 47]). Although valuations play a role in their definition, they
are not valuations themselves.

This article barely scratches the surface of a deep subject of which
much is known but even more is not. It is known, for example, that the
affine geometric invariants described here satisfy sharp affine isoperimetric inequalities that are stronger than the classical Euclidean geometric isoperimetric inequality. These inequalities also lead to sharp affine Sobolev inequalities that are stronger than sharp Euclidean Sobolev inequalities. It is also either known or conjectured that these same invariants, when restricted to convex bodies, satisfy sharp reverse affine isoperimetric inequalities, where the extremal bodies are simplices, in contrast to the sharp affine isoperimetric inequalities, where the extremal bodies are ellipsoids. See, for example, [19, 21, 26–28, 38, 43, 44, 53, 70, 71, 73–75, 79, 85, 86, 92, 93].

I am very grateful to Erwin Lutwak, Gaoyong Zhang, and Monika Ludwig not only for their invaluable help in writing this paper but also for teaching me everything I know about affine convex geometry. I would also like to give many thanks to Franz Schuster, Christoph Haberl and the referee for a careful reading of the original draft of this paper and many improvements.

2. Basic definitions and notation

We shall suppress the use of co-ordinates and therefore the space $\mathbb{R}^n$. Instead, we will always work with an $n$-dimensional real vector space denoted by $X$.

The vector space dual to $X$ will be denoted $X^*$. The natural contraction between a vector $x \in X$ and a dual vector $\xi \in X^*$ will be denoted by $\langle \xi, x \rangle = \langle x, \xi \rangle$.

We will fix a choice of Lebesgue measure $m$ on $X$ and denote the corresponding constant differential $n$-form $dm \in \Lambda^n X^*$. This, in turn, naturally induces a dual Lebesgue measure $m^*$ and corresponding differential $n$-form $dm^* \in \Lambda^n X$.

Given a differentiable function $f : X \to \mathbb{R}$, recall that its differential $\partial f : X \to X^*$ is defined by

$$\langle \partial f(x), v \rangle = \frac{d}{dt} \bigg|_{t=0} f(x + tv).$$

Since a vector space has a natural linear connection, we can also define, for a twice differentiable function $f$, the Hessian $\partial^2 f : X \to S^2 X^*$, where $S^2 X^*$ denotes the symmetric tensor product of $X^*$, by

$$\langle \partial^2 f(x), v \otimes w \rangle = \frac{d}{dt} \bigg|_{t=0} \partial f(x + tv)w.$$
the transpose of $g$ will be denoted by $g^t : X^* \to X^*$ and its inverse by $g^{-t}$.

In particular, the naturally induced action of $\text{GL}(X)$ on $X^*$ is given by $\xi \mapsto g^{-t}\xi$ for each $\xi \in X^*$ and $g \in \text{GL}(X)$. If $Y$ is a subspace of a tensor product of one or more copies of $X$ or $X^*$, then there is an action of $\text{GL}(X)$ on $Y$ naturally induced by the actions on $X$ and $X^*$.

The naturally induced action of $\text{GL}(X)$ on a function $f : X \to Y$ is $f \mapsto f_g$, where $g \in \text{GL}(X)$ and $f_g(x) = gf(g^{-1}x)$ for each $x \in X$. The corresponding induced action of $\text{GL}(X)$ on a function $f : X^* \to Y$ is $f \mapsto f_g$, where $f_g(\xi) = gf(g^t\xi)$ for each $\xi \in X^*$.

Each linear transformation $A : X \to X$ naturally induces a linear transformation $\det A : \Lambda^nX \to \Lambda^nX$, where $\Lambda^nX$ is the $n$-exterior power of $X$. Since $\Lambda^nX$ is 1-dimensional, $\det A \in \mathbb{R}$. Let $\text{SL}(X)$ or $\text{SL}(n)$ denote the group of linear transformations with $\det A = 1$. For each $d \in \mathbb{R}$, let $(\Lambda^nX)^d$ denote the $d$-th power of $\Lambda^nX$.

Let $S^2_+X$ denote the space of positive definite symmetric tensor product of $X$ and $S^2_+X$ the space of nonnegative definite tensors. Each $A \in S^2X$ can be viewed as a linear transformation $A : X^* \to X$, and its determinant is a linear map $\det A : \Lambda^nX^* \to \Lambda^nX$ and therefore $\det A \in (\Lambda^nX)^2$.

3. Objects of study

3.1. Geometric setting. Although we will be studying geometric objects in $\mathbb{R}^n$, we want to suppress the appearance of co-ordinates.

Therefore, throughout this paper, we will denote by $X$ an $n$-dimensional vector space and $X^*$ its dual space. Note that $X$ can be viewed as flat affine space with a distinguished point. Also, note that there is a natural isomorphism of the tangent space $T_xX \simeq X$ for each $x \in X$.

3.2. Convex body. A body is a compact set $K$ in $X$ or $X^*$ that is the closure of its interior. We denote its boundary by $\partial K$. A convex body is a body that is convex.

For convenience we will assume in the rest of this paper that $K \subset X$ is a convex body that contains the origin in its interior.

3.3. The space of all convex bodies. A convex body in a vector space $X$ is a compact convex set with nonempty interior. The set of all such bodies will be denoted $\mathcal{C}(X)$. This space is a complete metric space with respect to Hausdorff distance. It is also an Abelian semigroup under the operation of Minkowski sum, which is defined by

$$K + L = \{x + y : x \in K, \ y \in L\}.$$
We will often restrict to convex bodies that contain the origin in their interiors. The set of all such bodies will be denoted $\mathcal{C}_0(X)$.

We also denote by $\mathcal{C}_2^0(X)$ the set of convex bodies in $\mathcal{C}_0(X)$ with support functions that are twice continuously differentiable (see §5.1 for the definition of a support function).

The ultimate goal is to study the affine geometric properties of convex bodies; this is equivalent to studying the affine geometric properties of compact convex hypersurfaces. Here, we will focus on geometric properties that are invariant only under affine transformations that fix a point (known as the origin). Such transformations are sometimes called centro-affine transformations, but we prefer the more commonly used linear transformations.

3.4. Valuations. We need an abstract framework for the study and classification of integral invariants of convex bodies. The appropriate setting is that of valuations, which we define here.

Given a set $S \subset \mathcal{C}(X)$, a valuation on $S$ is a finitely additive function $\Phi : S \rightarrow \mathbb{R}$. In other words, if $A, B, A \cap B, A \cup B \in S$, then

$$\Phi(A) + \Phi(B) = \Phi(A \cup B) + \Phi(A \cap B).$$

More generally, if $Y$ is an additive semigroup, then a $Y$-valued valuation is a function $\Phi : S \rightarrow Y$ satisfying (1).

If $G$ is a group acting on $X$ and $S \subset \mathcal{C}(X)$ is invariant under the action of $G$, then a valuation $\Phi : S \rightarrow \mathbb{R}$ is $G$-invariant, if $\Phi(gK) = \Phi(K)$, for each $g \in G$ and $K \in S$.

If $G$ acts on vector spaces $X$ and $Y$, then a $Y$-valued valuation $\Phi : S \rightarrow Y$ is $G$-equivariant, if $\Phi(gK) = g\Phi(K)$, for each $g \in G$ and $K \in S$.

If $GL(X)$ acts on $Y$, then a valuation $\Phi : S \rightarrow Y$ is $GL(X)$-homogeneous of degree $d$, if

$$\Phi(gK) = |\det g|^{d/n} g\Phi(K),$$

for each $g \in GL(X)$ and $K \in S$.

Valuations has been and still are an active area of research. For recent work that will not be discussed here, including extensions from $\mathbb{R}^n$ to more general spaces such as Riemannian manifolds, see, for example, [2–18, 22–24, 33, 34, 48–52, 80–84], as well as the references in [63].

4. Overall strategy

The standard strategy for studying a geometric object is to classify and study the properties of simpler objects that are derived from the
original object and that behave appropriately under geometric transformations.

For example, differential geometers study submanifolds in Euclidean space usually by focusing on local differential invariants that are invariant under rigid motions. Invariant theory tells us that the only such invariants are the first and second fundamental forms and covariant derivatives of the second fundamental form (all covariant derivatives of the first fundamental form vanish).

We would like to follow the outline—if not the details—of such a strategy to study the affine geometry of convex bodies. In particular, our goal is to define and classify global affine invariant integral (instead of local differential) invariants of convex bodies.

The appropriate abstract setting for integral invariants turns out to be the study of continuous valuations on convex bodies that behave well under linear transformations, as defined above. These valuations lead naturally to affine integral invariants of a convex body. Recent classification theorems obtained by Ludwig and Reitzner show that all possible affine integral invariants are obtained by the constructions presented here.

5. Fundamental constructions

5.1. The support function. The support function of a set $K \subset X$ is the function $h_K : X^* \to \mathbb{R}$ given by

$$h_K(\xi) = \sup \{ \langle \xi, x \rangle : x \in K \}.$$ 

Observe that $h_K$ is convex and homogeneous of degree 1 and is strictly positive if the origin lies in the interior of the convex hull of $K$. Given $\xi \in X^*$, the level sets of $\xi : X \to \mathbb{R}$ comprise a family of parallel hyperplanes. The level set $\{ x : \langle \xi, x \rangle = h_K(\xi) \}$ is the “last” hyperplane in the family that touches $K$.

If $K \in \mathcal{C}(X)$, then $K$ can be recovered from the support function by

$$K = \{ x : \langle \xi, x \rangle \leq h_K(\xi), \xi \in X^* \}.$$ 

The support function behaves nicely under affine transformations. In particular, given an affine transformation $x \mapsto Ax + v$ for each $x \in X$, where $A : X \to X$ is a linear transformation and $v \in X$. Then

$$h_{AK+v}(\xi) = h_K(A^t\xi) + \langle \xi, v \rangle.$$ 

The support function is also a valuation.

**Lemma 5.1.** The map $K \mapsto h_K$ is a $GL(X)$-equivariant valuation from $\mathcal{C}(X)$ to the space of real-valued functions on $X^*$. 

Proof. If $K, L, K \cup L \in C_0(X)$, then
\[
\begin{align*}
    h_{K \cup L} &= \max(h_K, h_L) \\
    h_{K \cap L} &= \min(h_K, h_L).
\end{align*}
\]
Therefore,
\[
h_{K \cup L} + h_{K \cap L} = h_K + h_L.
\]
$GL(X)$-equivariance follows from (2). \qed

The same argument also implies the following for convex bodies in $C_0^2(X)$.

**Lemma 5.2.** For each function $\Phi : [0, \infty) \times X \times S^2_+ X \to Y$, where $Y$ is an additive semigroup, the map $K \mapsto \Phi_K$, where $\Phi_K : X^* \to Y$ is given by
\[
\Phi_K(\xi) = \Phi(h_K(\xi), \partial h_K(\xi), \partial^2 h_K(\xi)),
\]
is a valuation from $C_0^2(X)$ to the space of $Y$-valued functions on $X^*$.

If $GL(X)$ acts on $Y$ and there exists $q \in \mathbb{R}$ such that
\[
\Phi(t, gv, (g \otimes g) q) = (\det g)^q/n \Phi(t, v, q),
\]
for each $g \in GL(X)$ and $(t, v, q) \in [0, \infty) \times X \times S^2_+ X$, then $\Phi_K$ is a $GL(X)$-homogeneous valuation.

5.2. **The Minkowski sum.** Given sets $K, L \subset X$, we define their Minkowski sum to be
\[
K + L = \{x + y : x \in K, y \in L\}.
\]
If $K, L \in C(X)$, then the Minkowski sum is also given by
\[
h_{K+L} = h_K + h_L.
\]
By this and Lemma 5.1 it follows that the identity map $K \mapsto K$ is a $GL(X)$-equivariant valuation with respect to the Minkowski sum.

Firey [32] extended the notion of a Minkowski sum to that of an $L_p$ Minkowski sum. For $p \geq 1$, the $L_p$ Minkowski sum of two convex bodies is defined by
\[
h^p_{K + L} = h^p_K + h^p_L.
\]
By Lemma 5.2, the identity map $K \mapsto K$ is a $GL(X)$-equivariant valuation from $C_0(X)$ to $C_0(X)$ (with respect to the $L_p$ Minkowski sum).
5.3. The polar body. The polar body of $K$ is the convex body $K^* \subset X^*$ given by

$$K^* = \{ \xi : \langle \xi, x \rangle \leq 1 \text{ for each } x \in K \}.$$ 

Each $\xi \in X^*$ is in $K^*$ if and only if $K$ is contained in the half-space $\xi \leq 1$.

If $K \in \mathcal{C}_0(X)$, then the support functions $h_K : X \to \mathbb{R}$ and $h_{K^*} : X^* \to \mathbb{R}$ uniquely determine each other via the identities

$$h_{K^*}(x) = \sup_{\xi \in X^* \setminus \{0\}} \frac{\langle \xi, x \rangle}{h_K(\xi)},$$

$$h_K(\xi) = \sup_{x \in X \setminus \{0\}} \frac{\langle \xi, x \rangle}{h_{K^*}(x)}.$$ 

From this it follows that

$$K = \{ x : h_{K^*}(x) \leq 1 \}$$

$$K^* = \{ \xi : h_K(\xi) \leq 1 \}.$$ 

If $K$ is symmetric about the origin, then $h_{K^*}$ is simply the norm on $X$ with unit ball $K$ and $h_K$ is the dual norm with unit ball $K^*$. The support function and polar body therefore extend the concepts of a norm and its unit ball to asymmetric convex bodies.

Observe that if $K, L, K \cup L \in \mathcal{C}_0(X)$, then,

$$(K \cup L)^* = K^* \cap L^*.$$ 

Therefore,

Lemma 5.3. If $S^* \subset \mathcal{C}_0(X^*)$ and $\Phi : S^* \to Y$ is a valuation, then so is $\Phi^* : S \to Y$, where $\Phi^*(K) = \Phi(K^*)$, for each $K \in S$, and $S = \{ K : K^* \in S^* \}$.

Moreover, if $\Phi$ is GL($X^*$)-homogeneous, then $\Phi$ is GL($X$)-homogeneous.

5.4. The inverse Gauss map. If $K \in \mathcal{C}_0(X)$, then its support function $h = h_K$ is a convex function and therefore its differential is $L^\infty$ and defined almost everywhere. We observe here that, where the differential of $h$ is defined, it defines an affine invariant notion of the inverse Gauss map for the boundary of $K$.

It follows by the homogeneity of $h$ that $\langle \xi, \partial h(\xi) \rangle = h(\xi)$ for each $\xi \in X^*$. This implies that $\partial h(\xi) \in \partial K^*$ for each $\xi \in X^* \setminus \{0\}$. Moreover, $\xi$ is conormal to the boundary of $K$ at $\partial h(\xi)$; in other words, $\xi^\perp = \mathcal{T}_xK$, where $x = \partial h(\xi)$. For this reason we call the map $\partial h : X^* \setminus \{0\} \to \partial K$ the inverse Gauss map.

The precise relationship between $\partial h_K$ and the classical Gauss map is given as follows.
Proposition 5.4. If $X$ has an inner product, so $X^*$ can be identified with $X$, and $K \in C_0(X)$ has a differentiable support function $h_K$, then the Gauss map $\gamma_K : \partial K \to S^{n-1}$ is given by

$$\gamma_K(\partial h(\xi)) = \frac{\xi}{|\xi|},$$

for each $\xi \in X^* \setminus \{0\}$, where $|\xi|$ denotes the norm induced by the inner product.

5.5. The second fundamental form. Let $K \in C_0^2(X)$ and $h = h_K$. If $\partial h$ is the inverse Gauss map, then $\partial^2 h$ should be the inverse second fundamental form. That is indeed the case. First, observe that if $K \in C_0^2(X)$, then for each $\xi \in X^* \setminus \{0\}$, $T_x K = \xi^\perp$, where $x = \partial h(\xi)$.

Since $h$ is a convex function, $\partial^2 h$ is nonnegative definite. Moreover, since $\partial h$ is homogeneous of degree 0, $\langle \xi, \partial^2 h(\xi) \rangle = 0$, and therefore $\partial^2 h(\xi) \in S^2 \xi^\perp$. If $\partial^2 h(\xi)|_{\xi^\perp}$ is positive definite, then the second fundamental form of $\partial K$ at $x = \partial h(\xi)$ is given by

$$\Pi(x) = \left[ \partial^2 h(\xi)|_{\xi^\perp} \right]^{-1} \in [S^2 \xi^\perp]^* = S^2(X/R\xi) = S^2 T^*_x \partial K.$$

Observe that the definition of $\Pi$ is invariant under translations and therefore is an affine invariant. Therefore, the standard second fundamental form of a sufficiently smooth strictly convex hypersurface $\partial K$ is well-defined in an affine setting, even though the first fundamental form and Gauss map are not.

5.6. The Legendre transform. The inverse Gauss map can be made more useful, if it is made homogeneous of degree 1. This is best done by introducing the following variant of the support function.

Let $H = \frac{1}{2} h^2_K$. The differential of $H$, $\partial H : X^* \to X$, is homogeneous of degree 1. Its Hessian, $\partial^2 H : X^* \to S^2 X$ is homogeneous of degree 0 and always nonnegative definite.

If $K \in C_0^2(X)$, then the map $\partial H : X^* \setminus \{0\} \to X \setminus \{0\}$ is a $C^1$ diffeomorphism. We call this map the Legendre transform. It has the following very nice properties.

Lemma 5.5. If $K \in C_0^2(X)$, $K^* \in C_0^2(X^*)$, $H = \frac{1}{2} h^2_K$, $H^* = \frac{1}{2} h^2_{K^*}$, and

$$x = \partial H(\xi),$$

(6)
then the following hold:

(7) \quad h_{K^*}(x) = h_K(\xi).

(8) \quad \langle \xi, x \rangle = h_K(\xi)h_{K^*}(x)

(9) \quad \xi = \partial H^*(x)

(10) \quad \partial^2 H^*(x) = [\partial^2 H(\xi)]^{-1}

In particular, \partial^2 H(\xi) and \partial^2 H^*(x) are positive definite.

Proof. Denote \(h = h_K\) and \(h^* = h_{K^*}\).

We begin by showing that \(x \in X \setminus \{0\}\) and \(\xi \in X^* \setminus \{0\}\) satisfy (8) if and only if

(11) \quad x = h^*(x)\partial h(\xi).

First, equation (8) implies that the maximum is achieved in the right side of (3), and therefore \(\xi\) is a critical point of the right side of (3). This implies (11). Conversely, if \(x\) and \(\xi\) satisfy (11), then by the homogeneity of \(h\),

\[
\langle \xi, x \rangle = \langle \xi, h^*(x)\partial h(\xi) \rangle \\
= h^*(x)\langle \xi, \partial h(\xi) \rangle \\
= h^*(x)h(\xi).
\]

Also, by (11) and the homogeneity of \(h^*\),

\[
h^*(x) = h^*(h^*(x)\partial h(\xi)) = h^*(x)h^*(\partial h(\xi)).
\]

It follows that \(h^*(\partial h(\xi)) = 1\). If we now set \(x = \partial H(\xi) = h(\xi)\partial h(\xi)\), then

\[
h^*(x) = h(h(\xi)\partial h(\xi)) = h(\xi)h(\partial h(\xi)) = h(\xi),
\]

establishing (7).

Equation (9) follows by switching the roles of \(\xi\) and \(x\) in the proof above.

Equation (10) and the positive definiteness of \(\partial^2 H\) and \(\partial^2 H^*\) follow by differentiating each side of (9) with respect to \(\xi\).

5.7. The curvature function. Next, we want to define an affine invariant curvature function, roughly equivalent to Gauss curvature. The idea is to measure the nonlinearity of \(h\) using the determinant of the Hessian of the support function \(h_K\). This, however, does not work, because the Hessian is singular. This can be seen by observing that the differential \(\partial h_K\) is homogeneous of degree 0 and therefore, by the Euler equation (14), \(\langle \xi, \partial^2 h(\xi) \rangle = 0\).

This awkward situation is best circumvented by using the function \(H\) defined above. Assume \(K \in C^2_0(X)\). Since the differential form \(dm\)
on $X$ is homogeneous of degree $n$, it follows by Lemma 6.1 (stated below) that $(\partial H)^*dm$ is an $n$-form homogeneous of degree $n$ on $X^*$. We can therefore write

\begin{equation}
(\partial H)^*dm = (\det \partial^2 H)dm^*,
\end{equation}

where the determinant is defined relative to the measures $dm$ and $dm^*$. Following standard practice in convex geometry, we define the curvature function of $K$ to be the function $f_K : X^* \to \mathbb{R}$ homogeneous of degree $-n - 1$ such that

\begin{equation}
(\det \partial^2 H) = h_{K}^{n+1} f_K.
\end{equation}

Since we have already shown that $\partial h_K$ is essentially the inverse Gauss map, it should not be a surprise that the curvature function, which is obtained in spirit by taking the determinant of the Hessian of the support function, is essentially the Gauss curvature of $\partial K$. More precisely, we have the following.

**Proposition 5.6.** If an inner product is put on $X$, giving it a Euclidean geometric structure, then the Gauss curvature $\kappa : \partial K \to \mathbb{R}$ of $\partial K$ is given by

\[ \kappa(\partial h_K(\xi)) = \frac{|\xi|^{-n-1}}{f_K(\xi)} \]

for each $\xi \in X^* \setminus \{0\}$.

This proposition can be proved by choosing orthonormal coordinates on $X$ such that $\xi$ is equal to the last basis vector and representing $\partial K$ locally as a graph over the hyperplane spanned by all but the last basis vector.

6. THE HOMOGENEOUS CONTOUR INTEGRAL

Our description of valuations relies on the calculus of homogeneous functions and, in particular, a special type of integral, which we call the homogeneous contour integral, because, as for a contour integral of a meromorphic function over a closed curve in the complex plane (minus the poles), the value of the integral depends only on the homology class of the contour. The reason is also the same: the integrand is really a closed form, so the value of the integral depends only on the cohomology class represented by the form (we learned this interpretation of the homogeneous contour integral from Juan Carlos Alvarez-Paiva).
6.1. **Homogeneous functions and differential forms.** Given $t > 0$, define $D_t : X \to X$ by $D_t(x) = tx$. A function $f : X \to \mathbb{R}$ is *homogeneous of degree $e$* if

$$f \circ D_t = t^e f.$$ 

Recall that a function $f$ is homogeneous of degree $e$ if and only if it satisfies the Euler equation,

$$\langle df(x), x \rangle = ef(x). \quad (14)$$

A differential form or measure $\mu$ on $X$ is *homogeneous of degree $e$*, if

$$D_t^* \mu = t^e \mu. \quad (15)$$

We will need the following:

**Lemma 6.1.** Let $X$ and $Y$ be vector spaces, $\mu$ a differential form or measure homogeneous of degree $e$ on $Y$, and $F : X \to Y$ a differentiable map homogeneous of degree 1. Then the differential form or measure $F^* \mu$ is homogeneous of degree $e$ on $X$.

6.2. **The homogeneous contour integral for a differential form.**

By differentiating (15) with respect to $t$ at $t = 0$, we see that if $\omega$ is an $n$-form homogeneous of degree 0 on $X$, then

$$d(x]\omega) = 0, \quad (16)$$

where $x]\omega$ denotes the interior product of the vector field $x$ with the $n$-form $\omega$, i.e. the $(n - 1)$-form such that given $n - 1$ vector fields $V_2, \ldots, V_n$,

$$(x]\omega)(V_2, \ldots, V_n) = \omega(X, V_2, \ldots, V_n).$$

If an orientation is chosen for $X$, then we can define the *homogeneous contour integral of $\omega$* to be

$$\oint_X \omega = \int_{\partial D} x]\omega, \quad (17)$$

where $D$ is any bounded domain with the origin in its interior and smooth boundary $\partial D$. By (16), this is well-defined and independent of the domain $D$.

6.3. **The homogeneous contour integral for a measure.** A measure $\mu$ on $X$ is homogeneous of degree $e$ if for each $t > 0$, $D_t^* \mu = t^e \mu$ or, equivalently, for each compactly supported continuous function $f : X \to \mathbb{R}$,

$$\int f(tx) \, d\mu(x) = t^{-e} \int f(x) \, d\mu(x).$$

We describe below how to extend the definition (17) of a homogeneous contour integral of a homogeneous differential form of degree 0 to that
of a homogeneous contour integral of a homogeneous measure of degree 0.

We call $\Omega \subset X \setminus \{0\}$ a star domain, if there is a positive continuous function $h : X \setminus \{0\} \to (0, \infty)$ homogeneous of degree 1 such that
$$\Omega = \{ x : h(x) \leq 1 \}.$$ 

For each $\lambda > 1$, denote
$$R_\lambda \Omega = \lambda \Omega \setminus \Omega = \{ x : 1 < h(x) \leq \lambda \}.$$ 

Lemma 6.2. If $\mu$ is a measure homogeneous of degree 0 on $X \setminus \{0\}$ and $\lambda > 1$, then the quantity
$$\frac{\mu(R_\lambda \Omega)}{\log \lambda}$$
is independent of both $\lambda > 1$ and the star domain $\Omega$.

Proof. Fix $\lambda > 1$. Let $\Omega$ and $\Omega'$ be star domains in $X$. Choose $t > 0$ so that $\lambda t \Omega' \subset \Omega$. Therefore,
$$t \Omega' \subset \lambda t \Omega' \subset \Omega \subset \lambda \Omega.$$ 

Since $\mu$ is homogeneous of degree 0,
$$0 = \mu(\lambda \Omega \setminus \lambda t \Omega') - \mu(\Omega \setminus t \Omega')$$

$$= \mu(\lambda \Omega \setminus \Omega) + \mu(\Omega \setminus \lambda t \Omega') - \mu(\Omega \setminus \lambda t \Omega') - \mu(\lambda t \Omega' \setminus t \Omega')$$

$$= \mu(R_\lambda \Omega) - \mu(R_\lambda t \Omega')$$

$$= \mu(R_\lambda \Omega) - \mu(R_\lambda \Omega').$$

Therefore, the quantity (18) is independent of the star domain $\Omega$.

The lemma now follows by observing that the function $f : (0, \infty) \to (0, \infty)$ given by
$$f(t) = \mu(R_t \Omega)$$
is a monotone function satisfying the Cauchy functional equation
$$f(s + t) = f(s) + f(t).$$
and therefore (see, for example, [1]) $f(t) = ct$ for each $t \in (0, \infty)$ and a constant $c$. \qed

We can therefore define the homogeneous contour integral of a measure $\mu$ homogeneous of degree 0 on $X$ to be
$$\oint_X \mu = \frac{\mu(R_\lambda \Omega)}{\log \lambda},$$
where $\lambda > 1$ and $\Omega \subset X$ is a star domain.
The theorem below gives yet another formula for the homogeneous contour integral and establishes the equivalence of the different formulas.

**Theorem 6.3.** If $h : X \to [0, \infty)$ is a continuous and homogeneous of degree 1 and $\chi : (0, \infty) \to \mathbb{R}$ is a measurable function such that

$$\int_0^\infty \chi(t) \frac{dt}{t} < \infty,$$

then for each measure $\mu$ homogeneous of degree 0 on $X$,

$$\oint_X d\mu = \left( \int_X \chi(h(x)) d\mu(x) \right) \bigg/ \left( \int_0^\infty \chi(t) \frac{dt}{t} \right).$$

(20)

In particular, the value of the right side is independent of $h$ and $\chi$.

If the measure $\mu$ can be written as $d\mu = f dm$, where the function $f : X \backslash \{0\} \to [0, \infty)$ is continuous (and homogeneous of degree $-n$) and $D \subset X$ is a compact domain with smooth boundary and the origin in its interior, then

$$\oint_X d\mu = \int_{\partial D} x \cdot d\mu.$$  

(21)

**Remark.** If we fix an inner product on $X$ and in (21) let $\Omega$ be the Euclidean unit ball, then we obtain the formula commonly seen in convex geometry,

$$\oint_X f dm = \int_{S^{n-1}} f(\theta) d\Theta,$$

(22)

where $S^{n-1}$ is the unit sphere.

**Proof.** We provide two different approaches to proving the theorem. If the measure $\omega$ is absolutely continuous with respect to Lebesgue measure on $X \backslash \{0\}$, then straightforward proofs can be obtained by fixing an inner product and corresponding Euclidean structure on $X$ and using polar co-ordinates. We also provide a proof that do not rely on an inner product.

Suppose

$$d\mu = f dm,$$

where $f : X \backslash \{0\} \to \mathbb{R}$ is continuous and homogeneous of degree $-n$.

Fix an inner product on $X$ and define polar co-ordinates $(r, \theta) : X \to [0, \infty) \times S^{n-1}$, where $x = r\theta$, $r = |x|$, $\theta = x/|x|$, and $S^{n-1}$ is the standard unit sphere. Rescaling the inner product if necessary, we can write $dm = r^{n-1} dr \, d\Theta$, where $\Theta$ is the standard volume measure on $S^{n-1}$. 
Then
\[
\int_X \chi(h(x)) f(x) \, dm(x) = \int_0^\infty \int_{S^{n-1}} \chi(rh(\theta)) r^{-n} f(\theta) r^{n-1} \, d\Theta \, dr
\]
\[
= \int_{S^{n-1}} \left( \int_0^\infty \chi(rh(\theta)) r^{-1} \, dr \right) f(\theta) \, d\Theta
\]
\[
= \left( \int_0^\infty \chi(t) t^{-1} \, dt \right) \int_{S^{n-1}} f(\theta) \, d\Theta.
\]
This proves that the right side of (20) does not depend on the function \( \chi \) and by (22) is equal to (21). Equation (20) itself now follows by setting \( h \) equal to the homogeneous function of degree 1 such that \( \Omega = \{h \leq 1\} \) and \( \chi(t) \) equal to 1 if \( 1 \leq t \leq \lambda \) and 0 otherwise.

We can, however, prove (20) in full generality without relying on a Euclidean structure. It suffices to do this when \( \chi \) is a piecewise constant function. Assume that there exist \( t_0 < t_1 < \cdots < t_N < \infty \) and \( a_1, \ldots, a_N > 0 \) such that \( \chi(t) = a_k \) if \( t_{k-1} \leq t < t_k \) for some \( k \geq 1 \) and 0 otherwise. Then
\[
\int_X \chi(h(x)) \, d\mu(x) = \sum_{k=1}^N a_k \mu(t_{k-1} R_{t_k/t_{k-1}}(\Omega))
\]
\[
= \left( \int_X \, d\mu \right) \sum_{k=1}^N a_k (\log t_k - \log t_{k-1})
\]
\[
= \left( \int_X \, d\mu \right) \int_0^\infty \chi(t) \frac{dt}{t}.
\]

Last, we give a proof of (21) without relying on an inner product on \( X \). Fix \( \lambda > 1 \) and a continuous function \( h : X \setminus \{0\} \to (0, \infty) \) homogeneous of degree 1. Let \( \Omega = \{h \leq 1\} \) be the corresponding star domain. For each \( t \in (0, 1) \), we define a diffeomorphism \( \Phi_t : R_{\lambda/t} \Omega \to R_{\lambda/t} \Omega \) by
\[
\Phi_t(x) = \left[ \frac{\lambda - h(x)}{\lambda - 1} t + \frac{h(x) - 1}{\lambda - 1} \right] x.
\]
By equation (19),
\[
\mu(R_{\lambda/t} \Omega) = (\log \lambda - \log t) \int_X \, d\mu.
\]
Differentiating both sides with respect to \( t \) at \( t = 1 \), it follows that
\[
\frac{d}{dt} \bigg|_{t=1} \mu(R_{\lambda/t} \Omega) = - \int_X \, d\mu.
\]
On the other hand, if \( v : X \to X \) is given by

\[
v(x) = \frac{d}{dt} \bigg|_{t=1} \Phi_t(x) = \frac{\lambda - h(x)}{\lambda - 1} x
\]

and \( L_v \) denotes the Lie derivative with respect to the vector field \( v \), then by (24),

\[
\frac{d}{dt} \bigg|_{t=1} \mu(R_{\lambda/t}\Omega) = \frac{d}{dt} \bigg|_{t=1} \int_{R_{\lambda}\Omega} \Phi_t^* d\mu
\]

\[
= \int_{R_{\lambda}\Omega} L_v d\mu
\]

\[
= \int_{R_{\lambda}\Omega} d(v|d\mu)
\]

\[
= \int_{\lambda\partial\Omega} v|d\mu - \int_{\partial\Omega} v|d\mu
\]

\[
= -\int_{\partial\Omega} x|d\mu.
\]

By (23) and (25), the homogeneous contour integral defined by (19) is equal to (21) with \( D \) equal to any star domain \( \Omega \). Since \( d(x|d\mu) = 0 \), it follows that equality also holds for any domain \( D \) that contains the origin in its interior. □

Finally, the following shows that any standard integral of a homogeneous measure over a star domain can be written as an homogeneous contour integral.

**Lemma 6.4.** If \( h : X \to (0, \infty) \) is continuous and homogeneous of degree 1 and \( \mu \) is a measure on \( X\setminus\{0\} \) homogeneous of degree \( e \in (0, \infty) \), then

\[
\mu(\Omega) = \frac{1}{e} \int_X h^{-e} d\mu,
\]

where \( \Omega = \{x : h(x) \leq 1\} \).

**Proof.** Let \( \chi : [0, \infty) \to [0, \infty) \) be given by

\[
\chi(t) = \begin{cases} 
  t^e & \text{if } 0 \leq t \leq 1 \\
  0 & \text{otherwise},
\end{cases}
\]
for each $t \in [0, \infty)$. By (20),
\[
\mu(\Omega) = \int_{\Omega} d\mu = \frac{1}{e} \left( \int_X \chi(h(x))h^{-e}(x) \, d\mu(x) \right) / \left( \int_0^\infty \chi(t)t^{-1} \, dt \right) = \frac{1}{e} \oint h^{-e} \, d\mu.
\]

\[\square\]

6.4. Homogeneous integral calculus. A remarkable fact is that all of the standard formulas of integral calculus also hold for the homogeneous contour integral. The following results can be established using their standard counterparts and (20).

The following is a homogeneous version of Stoke’s theorem.

**Proposition 6.5.** If $\eta$ is an $(n-1)$-form homogeneous of degree 0, then $d\eta$ is an $n$-form homogeneous of degree 0 and
\[
\oint d\eta = 0.
\]

We can therefore integrate by parts.

**Corollary 6.6.** If $f$ and $g$ are functions on $X$ homogeneous of degree $d$ and $-n - d + 1$ respectively, then
\[
\oint_X f(x) \partial g(x) \, dm(x) = -\oint_X g(x) \partial f(x) \, dm(x).
\]

The following homogeneous version of the change of variables theorem also follows directly from the definition of an homogeneous contour integral and the standard change of variables formula.

**Proposition 6.7.** If $X$ and $Y$ are oriented $n$-dimensional vector spaces, $F : X \setminus \{0\} \to Y \setminus \{0\}$ is an orientation-preserving diffeomorphism homogeneous of degree 1, and $\omega$ is an $n$-form homogeneous of degree 0 on $Y$, then $F^* \omega$ is an $n$-form homogeneous of degree 0 on $X$ and
\[
\oint_X F^* \omega = \oint_Y \omega.
\]

**Corollary 6.8.** If $X$, $Y$, and $F$ are as defined in Proposition 6.7, $m_X$ and $m_Y$ are Lebesgue measures on $X$ and $Y$ respectively, and $g : Y \to \mathbb{R}$ is a measurable function homogeneous of degree $-n$, then
\[
\oint_Y g(y) \, dm_Y(y) = \oint_X g(F(x)) \det \partial F(x) \, dm_X(x).
\]
The following is a homogeneous version of the Riesz representation theorem.

**Proposition 6.9.** Given a bounded linear functional $\ell$ on the space of continuous functions homogeneous of degree $d$ on $X$, there exists a unique measure $\mu$ homogeneous of degree $-d$ on $X$ such that

$$\ell(f) = \oint f(x) \, d\mu(x).$$

7. **An explicit construction of valuations**

In this section we describe how to construct integral invariants of a convex body $K \subset X$ that behave well under $\text{GL}(X)$ and show that these invariants are valuations.

We begin with the obvious.

**Lemma 7.1.** If $K \mapsto \mu_K$ is a valuation from $C(X)$ to $Y$-valued measures homogeneous of degree 0 on a vector space $V$, where $Y$ is an additive semigroup, then so is $K \mapsto \oint_V \mu_K$.

The same is true, of course, for standard integrals, but we need only the homogeneous contour integral here.

By Lemma 7.1, Lemma 5.2, and Proposition 6.7,

**Proposition 7.2.** Given a vector space $Y$ and a measurable function $\phi : X^* \times (0, \infty) \times X \times S^2_+ X \to Y$ that is homogeneous of degree $-n$, the map $\Phi : C^2_0(X) \to Y$ given by

$$\phi(x, h_K(x), \partial h_K(x), \partial^2 h_K(x)) \, dm^*(x),$$

for each $K \in C^2_0(X)$, is a valuation.

If $\phi(g^t \xi, t, gv, gw) = \phi(\xi, t, v, w)$ for each $g \in \text{SL}(X)$, then $\Phi$ is an $\text{SL}(X)$-invariant valuation.

If there exists $d \in \mathbb{R}$ such that $\phi(g^t \xi, t, gv, gw) = (\det g)^d \, g\phi(\xi, t, v, w)$ for each $g \in \text{GL}(X)$, $\xi \in X^*$, $v \in X$, and $w \in S^2_+ X$, then $\Phi$ is a $\text{GL}(X)$-homogeneous valuation.

7.1. **Duality.**

**Proposition 7.3.** Given a function $\phi$ and valuation $\Phi$ as in Proposition 7.2, there is a dual function $\phi^*: X \times (0, \infty) \times X^* \times S^2_+ X^* \to Y$ homogeneous of degree $-n$ and corresponding valuation

$$\Phi^*(K^*) = \oint_X \phi^*(x, h_{K^*}(x), \partial h_{K^*}(x), \partial^2 h_{K^*}(x)) \, dm(x).$$
such that \( \Phi(K) = \Phi^*(K^*) \), for each \( K \in C^2_0(X) \cap [C^2_0(X^*)]^* \).

**Proof.** Assume that \( K \in C^2_0(X) \) and \( K^* \in C^2_0(X^*) \). The trick here is to write everything in terms of \( H = \frac{1}{2} h^2_K \).

Given a function \( \phi \) as defined in Proposition 7.2, define \( \psi : X^* \times (0, \infty) \times X \times S^2_+ X \to Y \) by

\[
\phi(\xi, t, x, q) = \psi \left( \xi, \frac{1}{2} t^2, tx, tq + x \otimes x \right).
\]

It follows by Lemma 5.5 that for each convex body \( K \in C^2_0(X) \) and \( \xi \in X^* , \)

\[
\psi(\xi, H_K(\xi), \partial H_K(\xi), \partial^2 H_K(\xi)) = \phi(\xi, h_K(\xi), \partial h_K(\xi), \partial^2 h_K(\xi)).
\]

Define \( \psi^* : X \times R \times X^* \times S^2_+ X^* \to Y \) by

\[
\psi^*(x, t, \xi, q) = \psi(\xi, t, x, q^{-1}),
\]

\( \phi^* : X \times R \times X^* \times S^2_+ X^* \to Y \) by

\[
\phi^*(x, t, \xi, q) = \psi^* \left( x, \frac{1}{2} t^2, t\xi, tq + \xi \otimes \xi \right),
\]

and

\[
\Phi^*(K^*) = \int_X \phi^*(x, h_{K^*}(x), \partial h_{K^*}(x), \partial^2 h_{K^*}(x)) dm(x).
\]

Then by Lemma 5.5 and Corollary 6.8, \( \Phi(K) = \Phi^*(K^*) \) for each \( K \in C^2_0(X) \), establishing the proposition. \( \square \)

7.2. **Volume.** The volume of a convex body and its polar are two examples of valuations of the type constructed above. Demonstrating this is perhaps most easily done using polar coordinates, but we insist on providing a proof that does not rely on an inner product.

Let \( V(\Omega) \) denote the volume of a set \( \Omega \subset X \) with respect to the Lebesgue measure \( m \). If \( \Omega \subset X^* \), then \( V(\Omega) \) denotes the volume with respect to the dual measure \( m^* \).

**Lemma 7.4.** If \( K \in C^2_0(X) \), then

\[
V(K) = \frac{1}{n} \int_X h^{\frac{-n}{2}}_{K^*} dm = \frac{1}{n} \int_{X^*} h_K f_K dm^*.
\]

**Proof.** The first equality follows from Lemma 6.4 with \( \mu = m \) and \( e = n \), and the second equality follows from the first by Corollary 6.8, (12), and (13). \( \square \)
8. Classification of valuations

We can now use Proposition 7.2 with canonical operations on the vector space $X$, its dual $X^*$, and symmetric tensors $S^2X$ to construct valuations that behave nicely under the action of $\text{GL}(X)$ and $\text{SL}(X)$. On the other hand, recent classification theorems by M. Ludwig and M. Reitzner show that this produces all possible such valuations.

9. Scalar valuations

9.1. $\text{SL}(n)$-invariant valuations. By Proposition 7.2 we can construct $\text{SL}(n)$-invariant valuations on $C^2_0(X)$ as follows.

**Corollary 9.1.** If $\phi : [0, \infty) \to [0, \infty)$ is continuous, then

$$\Omega_\phi(K) = \int_{X^*} \phi(h_{K f}^{n+1} h_{K}^{-n}) h_{K}^n \, dm^*$$

defines an $\text{SL}(n)$-invariant valuation on $C^2_0(X)$.

The valuation $\Omega_\phi$ is a generalization of affine surface area and $L_p$ affine surface area (see Section 9.1.1).

Using the Legendre transform, we get the following dual formula for the valuation $\Omega_\phi$ established by Ludwig [64], generalizing the theorem of Hug (see Section 9.2).

**Proposition 9.2.** If $\phi : [0, \infty) \to [0, \infty)$ is continuous, $\phi^* : [0, \infty) \to [0, \infty)$ is given by

$$\phi^*(t) = t \phi(1/t),$$

for each $t \in (0, \infty)$ and

$$\phi^*(0) = \lim_{t \to 0} t \phi(1/t),$$

then $\Omega_\phi(K) = \Omega_{\phi^*}(K^*)$ for each $K \in C^2_0(X)$.

**Proof.** By Proposition 7.3,

$$\Omega_\phi(K) = \int_{X^*} \phi(h_{K f}^{n+1} f_{K} h_{K}^{-n}) h_{K}^n \, dm^*$$

$$= \int_X \phi(1/(h_{K f}^{n+1} f_{K} f_{K^*})) h_{K f}^{n+1} f_{K} f_{K^*} h_{K}^{-n} \, dm$$

$$= \Omega_{\phi^*}(K^*),$$

□

To extend the valuation $\Omega_\phi$ to $C_0(X)$, the function $\phi$ must behave reasonably at 0 and $\infty$. M. Ludwig and M. Reitzner [66] have established the following converse.
Theorem 9.3. Any SL($X$)-invariant upper semicontinuous real-valued valuation on $C_0(X)$ that vanishes for polytopes must be equal to $\Omega_\phi$ for a concave function $\phi$ satisfying $\lim_{t \to 0} \phi(t) = \lim_{t \to \infty} \phi(t)/t = 0$.

9.1.1. GL($X$)-homogeneous valuations. An obvious way to construct scalar GL($X$)-homogeneous valuations on smooth convex bodies is as follows.

Corollary 9.4. For each $p \in (-\infty, -n) \cup (-n, \infty]$,

$$\Omega_p(K) = \int_{X^*} (h_K^{n+1} f_K)^{\frac{n}{n+p}} h_K^{-n} dm^*$$

defines a GL($X$)-homogeneous valuation on $C_0^2(X)$.

Note that $\Omega_0(K) = nV(K)$ and $\Omega_\infty(K) = nV(K^*)$.

The valuation $\Omega_1$ is known as affine surface area and was originally defined only for smooth convex bodies (see, for example, the monograph of Blaschke [25]). Different generalizations of affine surface area to arbitrary convex bodies were introduced by Leichtweiss [54], Schütt and Werner [96], and Lutwak [69]. The equivalence of these definitions was established by Schütt [95], Leichtweiss [55] (also, see Hug [45]), and Dolzmann-Hug [31].

Lutwak [72] introduced the more general invariant $\Omega_p$ as defined by (27), called it the $L_p$ affine surface area, and showed that it is invariant under SL($n$) transformations of the convex body. Lutwak also found a characterization of $\Omega_p$ that extends it to arbitrary and not just sufficiently smooth convex bodies. Hug [45] extended these results to $p > 0$, and Meyer and Werner [87] to $p > -n$.

Geometric interpretations of the $L_p$ affine surface area of a sufficiently smooth convex body that are valid for all $p \neq -n$ have been established by Schütt and Werner [97,98] and Werner and Ye [101].

M. Ludwig and M. Reitzner [66] have established the following converse to Corollary 9.4.

Theorem 9.5. Any upper semicontinuous real-valued valuation $\Phi$ on $C_0(X)$ that is GL($X$)-homogeneous of degree $q \in \mathbb{R}$ must be

$$\Phi = \begin{cases} b_0 \Omega_p & \text{if } q \in [-n, 0) \cup (0, n] \\ a_0 + b_0 \Omega_p & \text{if } q = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $a_0 \in \mathbb{R}$ and $b_0 \in [0, \infty)$ and

$$\frac{q}{n} = \frac{n - p}{n + p}.$$
9.2. **Hug’s theorem.** Hug [46] established a duality formula for the $L_p$ affine surface area of arbitrary convex bodies. For sufficiently smooth convex bodies, Hug’s theorem is easily proved using the Legendre transform.

**Proposition 9.6.** If $K \in \mathcal{C}_0^2(X)$ and $K^* \in \mathcal{C}_0^2(X^*)$, then

$$\Omega_p(K^*) = \Omega_{n^2/p}(K).$$

**Proof.** Set

$$\phi(t) = t^{p/(n+p)},$$

$$\phi^*(t) = t^{n/(n+p)}.$$

in Proposition 9.2. \qed

10. **Continuous GL$(n)$-homogeneous valuations**

In what follows, we will restrict our attention to valuations defined with the homogeneous contour integral, where the integrand is a function of a convex body’s support function and its derivative or those of the polar but without using second or higher derivatives. By theorems of M. Ludwig, it turns out that this is all that is needed in order to construct all possible continuous GL$(n)$-homogeneous valuations of convex bodies.

10.1. **Scalar valuations.** Of the scalar valuations defined in §9, the only ones that meet the restrictions above are the volume of the convex body and the volume of its polar. There is in fact another continuous scalar valuation, namely the constant valuation ($e(K) = 1$ for each nonempty convex body $K$), which cannot be written as an homogeneous contour integral.

Ludwig [59] established the following:

**Theorem 10.1.** A real-valued GL$(n)$-homogeneous continuous valuation on $\mathcal{C}_0(X)$ must be a constant times one of the following for each $K \in \mathcal{C}_0(X)$:

$$K \mapsto \begin{cases} 1 \\ V(K) \\ V(K^*) \end{cases}$$

10.2. **Vector-valued valuations.** Two obvious vector-valued linear invariants of a convex body are its center of mass and the center of mass of its polar. The center of mass of $K \in \mathcal{C}(X)$ is defined to be

$$c(K) = \frac{1}{V(K)} \int_K x \, dm(x).$$
The “denormalized” center of mass $V(K)c(K)$ is a valuation that has the following formulas.

**Proposition 10.2.** If $K \in C_0^2(X)$, then

\[
V(K)c(K) = \int_K x \, dm(x)
\]

(28) \hspace{1cm}
\[
= \frac{1}{n+1} \int_X x h_{K^*}^{-n-1}(x) \, dm(x)
\]

(29) \hspace{1cm}
\[
= \frac{1}{n+1} \int_{X^*} \partial h_K(\xi) h_K(\xi) f_K(\xi) \, dm^*(\xi)
\]

Observe that (28) extends the definition of $c(K)$ to each $K \in C(X)$, and (29) is valid for each $K \in C_0(X)$.

In dimension 2 there is an additional twist. Observe that $A \in \Lambda^2 X^*$ defines a linear map $A : X \to X^*$. Moreover, $g^t A g \in \Lambda^2 X^*$ for each $g \in \text{GL}(X)$. If $\dim X = 2$, then $\Lambda^2 X^*$ is 1-dimensional and therefore if $A$ is nonzero, then $g^t A g$ must be a scalar multiple of $A$. It is easily verified that if $\dim X = 2$, then

\[
g^t A g = (\det g) A,
\]

(30)

for each $A \in \Lambda^2 X^*$ and $g \in \text{GL}(X)$. This observation leads to the following:

**Lemma 10.3.** Assume $\dim X = 2$ and $S \subset C(X)$ is invariant under $\text{GL}(X)$.

If $\Phi : S \to X$ is a $\text{GL}(2)$-homogeneous valuation, and $A \in \Lambda^2 X^*$, then $A \Phi : S \to X^*$ is a $\text{GL}(2)$-homogeneous valuation.

If $\Phi : S \to X^*$ is a $\text{GL}(2)$-homogeneous valuation, and $A \in \Lambda^2 X$, then $A \Phi : S \to X$ is a $\text{GL}(2)$-homogeneous valuation.

**Proof.** We prove only the first statement. Assume $\Phi$ is $\text{GL}(2)$-homogeneous of degree $q \in \mathbb{R}$. If $g \in \text{GL}(X)$, then by the definition of $\text{GL}(n)$-homogeneity and (30),

\[
A \Phi(gK) = (\det g)^{q/2} A g \Phi(K)
\]

\[
= (\det g)^{q/2} (\det g) g^{-t} A \Phi(K)
\]

\[
= (\det g)^{1+q/2} g^{-t} A \Phi(K).
\]

$Ludwig$ [57] proved the following:
Theorem 10.4. If $\Phi : C_0(X) \to X$ is a nonvanishing $GL(n)$-homogeneous continuous valuation, then either there exists $a \in \mathbb{R}$ such that $\Phi(K) = aV(K)c(K)$ or $\dim X = 2$ and there exists $A \in \Lambda^2 X$ such that $\Phi(K) = V(K)Ac(K)$, for each $K \in C_0(X)$.

If $\Phi : C_0(X) \to X^*$ is a nonvanishing $GL(n)$-homogeneous continuous valuation, then either there exists $a \in \mathbb{R}$ such that $\Phi(K) = aV(K^*)c(K^*)$ or $\dim X = 2$ and there exists $A \in \Lambda^2 X^*$ such that $\Phi(K) = V(K)Ac(K)$, for each $K \in C_0(X)$.

There are obvious formulas for two other $GL(n)$-homogeneous vector-valued valuations, but by Corollary 6.8, they are identically zero in dimensions 2 and higher. One is

$$\int_{K^*} \partial h_K(\xi) \, dm^*(\xi) = \frac{1}{n} \oint h_K^{-n} \partial h_K \, dm^*(\xi)$$
$$= -\frac{1}{n(n-1)} \oint \partial (h_K^{-n+1})$$
$$= 0.$$  

The other is just the same formula applied to $K^*$.

11. Matrix-valued valuations

A natural symmetric matrix-valued linear invariant of $K \in C_0(X)$ is its second moment

$$M_2(K) = \frac{n + 2}{V(K)} \int_K x \otimes x \, dm(x) \in S^2 X.$$  

The normalization constant $n + 2$ is chosen so that if $K$ is the unit ball of the inner product $g \in S^2 X^*$, then $M_2(K) = g^{-1} \in S^2$.

A new symmetric matrix-valued linear invariant of $K \in C_0(X)$ was introduced in [74] (and later identified as being analogous to the Fisher information of a probability density). This is defined to be

$$M_{-2}(K) = \frac{n}{V(K)} \int_K \partial h_{K^*}(x) \otimes \partial h_{K^*}(x) \, dm(x) \in S^2 X^*.$$  

Note that $M_{-2}(K)$ is not necessarily equal to $M_2(K^*)$. Again, the normalization constant $n$ is chosen so that if $K$ is the unit ball of the inner product $g \in S^2 X^*$, then $M_{-2}(K) = g$.

The denormalized invariants are valuations given by the following formulas.
Proposition 11.1. If $K \in C_0^n(X)$, then

$$V(K)M_2(K) = (n + 2) \int_K x \otimes x \, dm(x)$$

$$= \int_X (x \otimes x) h_{K^*}^{-n-2}(x) \, dm(x)$$

$$= \int_{X^*} [\partial h_{K^*}(\xi) \otimes \partial h_{K^*}(\xi)] h_K(\xi) f_K(\xi) \, dm^*(\xi)$$

$$= n \int_{K^*} [\partial h_{K^*}(\xi) \otimes \partial h_{K^*}(\xi)] h_{K^*}^{n+1}(\xi) f_K(\xi) \, dm^*(\xi).$$

and

$$V(K)M_{-2}(K) = n \int_K \partial h_{K^*}(x) \otimes \partial h_{K^*}(x) \, dm(x)$$

$$= \int_X [\partial h_{K^*}(x) \otimes \partial h_{K^*}(x)] h_{K^*}^{-n}(x) \, dm(x)$$

$$= \int_{X^*} (\xi \otimes \xi) h_{K^*}^{-1}(\xi) f_K(\xi) \, dm(x)$$

$$= (n + 2) \int_{K^*} (\xi \otimes \xi) h_{K^*}^{n+1}(\xi) f_K(\xi) \, dm^*(\xi).$$

Equations (31) and (33) extend the definition of $M_2(K)$ and $M_{-2}(K)$ to each $K \in C(X)$, while (32) and (34) are valid for each $K \in C_0(X)$.

In [60] Ludwig classifies symmetric matrix-valued valuations. Ludwig [60] established the following:

Theorem 11.2. If $\Phi : C_0(X) \rightarrow S^2 X$ is a nonvanishing $GL(n)$-homogeneous continuous valuation, then there exists $a \in \mathbb{R}$ such that either $\Phi(K) = aV(K)M_2(K)$ or $\Phi(K) = aV(K^*)M_{-2}(K^*)$.

If $\Phi : C_0(X) \rightarrow S^2 X$ is a nonvanishing $GL(n)$-homogeneous continuous valuation, then there exists $a \in \mathbb{R}$ such that either $\Phi(K) = aV(K)M_{-2}(K)$ or $\Phi(K) = aV(K^*)M_2(K^*)$.

11.1. The Cramer-Rao inequality. The Cramer-Rao inequality [30, 91] states that the second moment matrix of a random vector dominates the inverse of the Fisher information matrix, and equality holds if and only if the random vector is Gaussian. The proof (see, for example, [29]) is elementary, requiring only the Cauchy-Schwarz inequality and integration by parts.

In [75] a Cramer-Rao inequality is proved for convex bodies. The proof given in [75] appears to be much more involved than the one for random vectors. We show here that it suffices to give essentially the
same proof as for random vectors but with all integrals replaced by homogeneous contour integrals.

First, we recall the original Cramer-Rao inequality and its proof.

**Theorem 11.3.** [30, 91] If $R$ is a random vector in $X$ with probability measure $f \, dm$, where $f : X \to [0, \infty)$ is differentiable and decays sufficiently rapidly at infinity, then

$$
E[R \otimes R] \geq E[\partial \log f(R) \otimes \partial \log f(R)]^{-1},
$$

with equality holding if and only if $R$ is Gaussian.

**Proof.** If $v \in X$ and $\xi \in X^*$, then by integration by parts and the Cauchy-Schwarz inequality,

$$
\langle \xi, v \rangle = \int_X \langle [v, \partial], \langle \xi, x \rangle f(x) \rangle \, dm(x)
$$

$$
= -\int_X \langle \xi, x \rangle \langle v, \partial f(x) \rangle \, dm(x)
$$

$$
= -\int_X \langle \xi, x \rangle \langle v, \partial \log f(x) \rangle f(x) \, dm(x)
$$

$$
\leq \left( \int_X \langle \xi, x \rangle^2 f(x) \, dm(x) \right)^{1/2} \left( \int_X \langle v, \partial \log f(x) \rangle^2 f(x) \, dm(x) \right)^{1/2}
$$

$$
= \left( \langle \xi \otimes \xi, f \rangle \langle v \otimes v, f \rangle \right)^{1/2}.
$$

Since this holds for all $v \in X$ and $\xi \in X^*$, inequality (35) follows. \qed

Next, we adapt the proof above to convex bodies.

**Theorem 11.4.** [75] If $K \in C_0(X)$, then

$$
M_2(K) \geq M_{-2}(K)^{-1},
$$

with equality holding if and only if $K$ is an ellipsoid centered at the origin.

**Proof.** If $K \in C_0(X)$, $v \in X$ and $\xi \in X^*$, then by Corollary 6.6 and the Cauchy-Schwarz inequality,

$$
\langle \xi, v \rangle = \frac{1}{nV(K)} \int_X \langle [v, \partial], \langle \xi, x \rangle \rangle h_{K^*}^{-n}(x) \, dm(x)
$$

$$
= -\frac{1}{V(K)} \int_X \langle \xi, x \rangle \langle v, \partial h_{K^*}(x) \rangle h_{K^*}^{-n-1}(x) \, dm(x)
$$

$$
\leq \left( \frac{1}{V(K)} \int_X \langle \xi, x \rangle^2 h_{K^*}^{-n-2}(x) \, dm(x) \right)^{1/2} \left( \frac{1}{V(K)} \int_X \langle v, \partial h_{K^*}(x) \rangle^2 h_{K^*}^{-n}(x) \, dm(x) \right)^{1/2}
$$

$$
= \langle \xi \otimes \xi, M_2(K) \rangle^{1/2} \langle v \otimes v, M_{-2}(K) \rangle^{1/2}.
$$
Since this holds for each $v \in X$ and $\xi \in X^*$, inequality (36) follows. \qed

12. Homogeneous function- and convex body-valued valuations

Observe that associated with $M_2(K)$ and $M_{-2}(K)$ are homogeneous functions $h_{\Gamma_2K} : X^* \rightarrow [0, \infty)$ given by

$$h_{\Gamma_2K}^2(\xi) = \langle \xi \otimes \xi, M_2(K) \rangle$$

and $h_{\Gamma^{-2}_2K} : X \rightarrow [0, \infty)$ given by

$$h_{\Gamma^{-2}_2K}^2(v) = \langle v \otimes v, M_{-2}(K) \rangle.$$  

The functions $h_{\Gamma_2K}$ and $h_{\Gamma^{-2}_2K}$ are in turn the support functions of ellipsoids, which we denote by $\Gamma_2K \subset X$ and $\Gamma^{-2}_2K \subset X^*$. These functions and bodies have the following natural generalizations.

If $p \in [1, \infty)$, let $\mathcal{H}_c^p(X)$ denote the space of convex real-valued functions on $X$ homogeneous of degree $p$. Note that a function $h : X^* \rightarrow \mathbb{R}$ is the support function of a convex body that contains the origin in its interior if and only if $h^p \in \mathcal{H}_c^p(X^*)$ and is positive outside the origin.

If $K \in \mathcal{C}_0(X)$, then a natural way to define convex homogeneous functions $h_{\Gamma_pK}^p \in H_c^p(X^*)$ and $h_{\Gamma_{-p}K}^p \in H_c^p(X^*)$ is by

$$h_{\Gamma_pK}^p(\xi) = \frac{c_{n,p}}{V(K)} \int_K |\langle \xi, x \rangle|^p dm(x)$$

and

$$h_{\Gamma_{-p}K}^p(v) = \frac{c_{n,p}}{V(K)} \int_K |\langle v, \partial H_{K^*}(x) \rangle|^p dm(x),$$

where

$$c_{n,p} = \frac{\omega_{2n}\omega_{-p-1}}{\omega_{n+p}},$$

and

$$\omega_k = \frac{\pi^{k/2}}{\Gamma(1 + \frac{k}{2})}$$

is the volume of the standard unit ball in $k$-dimensional Euclidean space. The normalization is again chosen so that if $K$ is the unit ball of an inner product, then $\Gamma_pK = \Gamma_{-p}K = K$.

Such functions are always even and can be generalized as follows. Given $p \in [1, \infty)$, $t \in [-1, 1]$, and $K \in \mathcal{C}_0^2(X)$, define convex bodies
\[ \Gamma_{-p}^t K, \Gamma_{-p}^t K \subset X \text{ such that} \]

\[ h_{\Gamma_{-p}^t K}^p (\xi) = \frac{c_{n,p}}{V(K)} \int_K [(1 + t)\langle \xi, x \rangle^p_+ + (1 - t)\langle \xi, x \rangle^p_-] \, dm(x) \]

\[ = \frac{c_{n,p}}{(n + p)V(K)} \int_X [(1 + t)\langle \xi, x \rangle^p_+ + (1 - t)\langle \xi, x \rangle^p_-] h_{K,-p}^{-n-p}(x) \, dm(x) \]

\[ = \frac{c_{n,p}}{(n + p)V(K)} \int_X [(1 + t)\langle \xi, \partial H_K(\eta) \rangle^p_+ + (1 - t)\langle \xi, \partial H_K(\eta) \rangle^p_-] h_{K}(\eta) f_K(\eta) \, dm^*(\eta) \]

\[ h_{\Gamma_{-p}^t K}^p (v) = \frac{c_{n,p}}{V(K)} \int_K [(1 + t)\langle v, \partial H_{K^*}(x) \rangle^p_+ + (1 - t)\langle v, \partial H_{K^*}(x) \rangle^p_-] \, dm(x) \]

\[ = \frac{c_{n,p}}{(n + p)V(K)} \int_X [(1 + t)\langle v, \partial H_{K^*}(x) \rangle^p_+ + (1 - t)\langle v, \partial H_{K^*}(x) \rangle^p_-] h_{K,-p}^{-n-p}(x) \, dm(x) \]

\[ = \frac{c_{n,p}}{(n + p)V(K)} \int_X [(1 + t)\langle v, \xi \rangle^p_+ + (1 - t)\langle v, \xi \rangle^p_-] h_{K}^{1-p}(\xi) f_K(\xi) \, dm^*(\xi) \]

where for each \( s \in \mathbb{R}, s_+ = s \) and \( s_- = 0 \) if \( s \geq 0 \) and \( s_+ = 0 \) and \( s_- = -s \) if \( s \leq 0 \).

Recent classification theorems of Ludwig [61], similar to the other theorems cited in this paper, suggest strongly the following conjecture of Ludwig: If \( n \geq 3 \), then any continuous GL\((n)\)-homogeneous \( \mathcal{H}_{c}^p(X^*) \)-valued valuation on \( C_0(X) \) is of the form

\[ aV(K)h_{\Gamma_{-p}^t K}^p \text{ or } aV(K^*)h_{\Gamma_{-p}^t K}^p \]

for some \( t \in [-1, 1] \) and \( a \in [0, \infty) \) and that any continuous GL\((n)\)-homogeneous \( \mathcal{H}_{c}^p(X) \)-valued valuation on \( C_0(X) \) is of the form

\[ aV(K)h_{\Gamma_{-p}^t K}^p \text{ or } aV(K^*)h_{\Gamma_{p}^t K}^p \]

for some \( t \in [-1, 1] \) and \( a \in [0, \infty) \).

13. QUESTIONS

We end with a few rather open-ended questions.

- The discussion here exploits Proposition 6.7 and Corollary 6.8, where the change of variables is always either a linear transformation of \( X \) or the Legendre transform associated with a convex body. Are there further applications of these results,
where more general diffeomorphisms homogeneous of degree 1 are used?

- Symmetric 2-tensor-valued valuations have been studied extensively. Do continuous GL(n)-homogeneous anti-symmetric 2-tensor-valued valuations exist? (Probably not)

- Another obvious way to define a continuous GL(n)-homogeneous matrix-valued valuation $\Phi : C_0^2(X) \to X \otimes X^*$ is

$$\Phi(K) = \frac{1}{n} \int_X x \otimes \partial h_{K^*}(x) h_{K^*}^{-1}(x) \, dm(x),$$

for each convex body $K$. The trace of this valuation is the volume of $K$. What if anything can be said about this valuation? Is there a classification of continuous GL(n)-homogeneous $X \otimes X^*$-valued valuations? The conjecture would be that the only possibilities are $K \mapsto \Phi(K)$ or $\Phi(K^*)$.

- The theorems of Ludwig on the classification of continuous GL(n)-homogeneous valuations are proved by analyzing the valuations when restricted to polytopes. Is there a different proof using the properties of the homogeneous contour integral?

- The approach taken here leads naturally to the definition of local geometric invariants of a Finsler manifold, by using the homogeneous contour integral on each tangent or cotangent space. For example, if $K_x \subset T_x M$ is the unit ball of a Finsler metric at a point $x$ in the manifold $M$, then $\Gamma^2 K_x$, $\Gamma^-2 K_x$, $\Gamma^2 K_x^*$, and $\Gamma^-2 K_x^*$ define different Riemannian metrics naturally associated with the Finsler metric. Are such invariants useful in Finsler geometry?

- It has been shown [76–78] that affine valuations of convex bodies are naturally associated with moments and generalized Fisher information (also known as Sobolev norms) of probability density functions. Can the concept of valuations be extended to probability densities or measures and classified in that context?

References


46. ______, *Curvature relations and affine surface area for a general convex body and its polar*, Results Math. 29 (1996), no. 3-4, 233–248. MR 1387565 (97c:52004)
64. ______, *General affine surface areas*, preprint, 2008.