And now for something completely different

Affine integral geometry from a differentiable viewpoint

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Overview

- Goal
  - A modern framework for affine integral geometry
- Outline
  - Euclidean geometry of a convex body
  - Constructing affine integral invariants of a convex body
    - Homogeneous contour integral
    - Homogeneous functions associated with a convex body
    - Constructing valuations
  - Classification theorems of M. Ludwig
  - Hug's theorem
  - Cramér-Rao inequality

Affine integral geometry

- Object of study
  - A convex body $K \subset \mathbf{R}^n$ is a convex set with non-empty interior
  - Often assume origin lies in interior of K
- Affine geometric invariants of K
- Sharp geometric inequalities
  - Generalized isoperimetric inequalities
- Ties to other fields
  - Functional analysis
  - Probability
  - Information theory
- Essentially equivalent to the study of "trivial" flat Finsler manifolds

Two known approaches

- Local affine differential geometry
    - Local differential geometry of a hypersurface in Euclidean space
    - Prolongation beyond second derivatives required
    - Works for only sufficiently smooth convex bodies
- Global affine integral geometry
    - What this talk is about

Euclidean geometry of a convex body $K \subset \mathbf{R}^n$

- Intrinsic geometry of boundary ∂K
  - Induced Riemannian metric g
  - Induced surface area measure dA
  - Gauss curvature κ

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Euclidean geometry of a convex body $K \subset \mathbf{R}^n$

- Local extrinsic geometry
  - Gauss map $\nu : \partial K \to S^{n-1}$
  - Second fundamental form $\mathrm{II} = \partial\nu : \partial K \to \mathrm{Sym}^2 \nu^\perp(x)$
  - Symmetric functions of principal curvatures (eigenvalues of II)
- Global extrinsic geometry
  - Volume of K
  - Surface area of K
  - Center of mass of K
  - Legendre-Binet ellipsoid (covariance metric of uniform distribution on K)
  - Integral of curvature function over K

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The Euclidean isoperimetric inequality

If B is the standard unit ball in $\mathbf{R}^n$, then

$$\frac{V_{n-1}(\partial K)}{V(K)^{(n-1)/n}} \geq \frac{V_{n-1}(\partial B)}{V(B)^{(n-1)/n}}$$

Equality holds if and only if K is a ball

Affine geometry of a convex body

- Desirable features
  - Invariant or equivariant under linear transformations
  - (optional) Invariant under translations
  - Works for arbitrary (not necessarily smooth) convex bodies
- Basic tools available
  - First and second derivatives of boundary
    - First derivative is L^∞ (boundary is Lipschitz)
    - Second derivative is a measure
  - Integration, i.e. averaging

Volume invariants

- Volume itself is not scale invariant and therefore not affine invariant
- But relative volume is
  - Given two convex bodies K and L, $V(K)/V(L)$ is invariant under affine transformations
- Another is the uniform probability measure on K,

$$d\mu = \frac{dx}{V(K)}$$

  - Used for averaging

Two known ways of constructing more affine invariants

- Optimize a scale invariant Euclidean invariant
- Affine average of a lower dimensional geometric invariant

An affine surface area for a convex body K via optimization

- For each inner product g on $\mathbf{R}^n$, let V_g be the volume of K and S_g be the surface area of ∂K with respect to g.
- Minimize $S_g / V_g^{(n-1)/n}$ over all inner products g.
- Euclidean isoperimetric inequality implies an affine isoperimetric inequality, where equality holds if and only if K is an ellipsoid
  - Nothing particularly new or interesting

Euclidean surface area equals average shadow area

- If $u \in S^{n-1}$, let $\pi_u : \mathbf{R}^n \to u^\perp$ denote orthogonal projection
- Area of shadow in direction u

$$V_{n-1}(\pi_u K) = \frac{1}{2}\int_{\partial K} |\nu(x) \cdot u| \, dA$$

- Average shadow area

$$\begin{aligned}
\int_{S^{n-1}} V_{n-1}(\pi_u K)\, du &= \frac{1}{2}\int_{S^{n-1}}\int_{\partial K} |\nu(x)\cdot u| \, dA\, du \\
&= \frac{1}{2}\int_{\partial K}\int_{S^{n-1}} |\nu(x)\cdot u| \, du\, dA \\
&= \left(\frac{1}{2}\int_{S^{n-1}} |e_n \cdot u| \, du\right)\int_{\partial K} dA \\
&= V(B^{n-1})V_{n-1}(\partial K)
\end{aligned}$$

Another affine surface area equals affine average of shadow area

- Euclidean surface area

$$S = \frac{1}{V_{n-1}(S^{n-1})} \int \left(\frac{1}{2} \int_{\partial K} |\nu(x) \cdot u| \, dA \right) \, du$$

- Affine surface area

$$A = \left(\frac{1}{V_{n-1}(S^{n-1})} \int \left(\frac{1}{2} \int_{\partial K} |\nu(x) \cdot u| \, dA \right)^{-n} \, du \right)^{-1/n}$$

But why is this affine invariant?

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Another affine isoperimetric inequality

- Affine surface area

$$A = \left(\frac{1}{V_{n-1}(S^{n-1})} \int \left(\frac{1}{2} \int_{\partial K} |\nu(x) \cdot u| \, dA \right)^{-n} du \right)^{-1/n}$$

- Another affine isoperimetric inequality

$$A \geq V^{(n-1)/n}$$

  Equality holds if and only if K is an ellipsoid
- Proved using Steiner symmetrization
- Implies the Euclidean isoperimetric inequality using Holder inequality
- Is much stronger than Euclidean inequality
- But why is $A/V^{(n-1)/n}$ affine invariant?

Constructing affine geometric invariants of a convex body

- Use homogeneous functions instead of tensors
- Use the **support function** and its derivatives
  - Its first derivative is essentially the Gauss map
  - Its second derivative is essentially the second fundamental form
- Define **valuations** using the **homogeneous contour integral**
  - With integrand equal to a function of the support function and its derivatives
- Invariants of a convex body are also invariants of its **polar**
  - Use **Legendre transform** to map between them
    - For scalar invariants this is a theorem of Hug
- Classification theorems of Ludwig, Reitzner, Schuster, Haberl, and others show that this construction gives all possible affine invariant valuations

Preliminaries

- For convenience, we fix an origin and a choice of Lebesgue measure on affine space
- Let X denote an n-dimensional vector space and dx the Lebesgue measure
- Let X^* denote the dual vector space and $d\xi$ the dual Lebesgue measure
- Both measures are homogeneous of degree n
- Let $\langle \xi, x \rangle \in \mathbf{R}$ denote the natural evaluation map for each $\xi \in X^*$ and $x \in X$
- Note that the identity map $x : X \to X$ is itself a vector field, sometimes written in co-ordinates as

$$x = x^i \frac{\partial}{\partial x^i}$$

Homogeneous functions and differential n-forms

- A function $h : X \to \mathbf{R}$ is homogeneous of degree d if $h(tx) = t^d h(x)$ for each $x \in X$ and $t > 0$
- A differential n-form μ on X is homogeneous of degree d if $D_t^* \mu = t^d \mu$ for each $t > 0$, where $D_t(x) = tx$ for each $x \in X$
- A differential n-form $\mu = m(x)\, dx$ is homogeneous of degree d if and only if m is homogeneous of degree $d - n$
- A differential n-form $\mu = m(x)\, dx$ is homogeneous of degree 0 if and only if

$$0 = \left.\frac{d}{dt}\right|_{t=0} D_t^* \mu = \mathcal{L}_x \mu = d(x \rfloor \mu)$$

The homogeneous contour integral

- If $\mu = m(x)\,dx$ is a differential n-form homogeneous of degree 0 on X, define the *homogeneous contour integral* of μ to be

$$\oint_X \mu = \int_{\partial\Omega} x \rfloor \mu = \int_{\partial\Omega} m(x)\, x \rfloor dx,$$

  where Ω is a bounded domain containing the origin in its interior.

- The value of this integral does not depend on the domain Ω.
  - If $\Omega' \subset\subset \Omega$,

$$\int_{\partial\Omega} x \rfloor \mu - \int_{\partial\Omega'} x \rfloor \mu = \int_{\Omega \setminus \Omega'} d(x \rfloor \mu) = 0.$$

- This integral was defined using only the natural linear operations on X and the Lebesgue measure dx. No inner product or norm on X was used at all.

Explicit formulas for the homogeneous contour integral

If m is homogeneous of degree $-n$, then

- if $X = \mathbf{R}^n$, Ω is the standard unit ball, S^{n-1} its boundary, and du the standard surface area measure on S^{n-1}, then

$$\oint_X m(x)\,dx = \int_{S^{n-1}} m(u)\,du,$$

- if h is homogeneous of degree 1 and $\chi : (0,\infty) \to [0,\infty)$ is continuous and compactly supported, then

$$\oint_X m(x)\,dx = \left(\int_X \chi(h(x)) m(x)\,dx \right) \Big/ \left(\int_0^\infty \chi(t) t^{-1}\,dt \right).$$

The value of the right side does not depend on the functions h and χ.

The homogeneous contour integral for a homogeneous measure

- A measure ω on $X\backslash\{0\}$ is *homogeneous of degree* 0, if for each measurable set $E \subset X\backslash\{0\}$ and $t > 0$, $\omega(tE) = \omega(E)$.
- The homogeneous contour integral of ω is defined to be

$$\oint_X \omega = \frac{\omega(\lambda\Omega\backslash\Omega)}{\log\lambda},$$

  where $\Omega \subset X$ is a star domain and $\lambda > 1$.
- The right side does not depend on either λ or Ω.

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Homogeneous vector calculus

The homogeneous contour integral satisfies the standard formulas of integral calculus.

- **Integration by parts.** If f is homogeneous of degree d and g is homogeneous of degree $-n-d+1$, then

$$\oint_X f(x)\partial g(x)\,dx = -\oint_X g(x)\partial f(x)\,dx.$$

- **Change of variables.** If X and Y are n-dimensional vector spaces, $\Phi : X \to Y$ a differentiable map homogeneous of degree 1, dx and dy Lebesgue measures on X and Y respectively, and $\psi : Y \to \mathbf{R}$ a homogeneous function of degree $-n$, then

$$\oint_Y \psi(y)\,dy = \oint_X \psi(\Phi(x)) \det \partial\Phi(x)\,dm_X(x).$$

Homogeneous functions associated with a convex body K

- The *polar support function* $h_K^*: X \to \mathbf{R}$

$$h_K^*(x) = \inf\left\{\lambda > 0 \ : \ \frac{x}{\lambda} \in K\right\}$$

- The *support function* $h_K : X^* \to \mathbf{R}$

$$h_K(\xi) = \sup\{\langle \xi, x\rangle \ : \ x \in K\}$$

- Both functions above are convex and homogeneous of degree 1
- Let $\phi_K = \frac{1}{2}h_K^2$ and $\phi_K^* = \frac{1}{2}(h_K^*)^2$
- Both ϕ_K and ϕ_K^* are convex and homogeneous of degree 2
- *We shall always assume that* $\phi_K : X^*\backslash\{0\} \to (0,\infty)$ *and* $\phi_K^* : X\backslash\{0\} \to (0,\infty)$ *are twice differentiable and have positive definite second derivatives*

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The affine Gauss map

- The map
$$\partial\phi_K^* = h_K^* \partial h_K^* : X \to X^*$$
is a diffeomorphism homogeneous of degree 1
- It is the affine analogue of the Gauss map
  - If $X = X^* = \mathbf{R}^n$, then the Gauss map of ∂K is given by
$$\nu(x) = \frac{\partial\phi_K^*(x)}{|\partial\phi_K^*(x)|}$$
for each $x \in \partial K$

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The affine inverse Gauss map

- The map
$$\partial\phi_K = h_K \partial h_K : X^* \to X$$
is a diffeomorphism homogeneous of degree 1
- It is the affine analogue of the inverse Gauss map
  - If $X = X^* = \mathbf{R}^n$, then the inverse Gauss map of ∂K is given by
$$\nu^{-1}(u) = \partial h_K(u) = \frac{\partial\phi_K(u)}{h_K(u)}$$
for each $u \in S^{n-1}$
- $\partial\phi_K$ and $\partial\phi_K^*$ are inverse maps
  - $\partial\phi_K(\partial\phi_K^*(x)) = x$ for each $x \in X$
  - $\partial\phi_K^*(\partial\phi_K(\xi)) = \xi$ for each $\xi \in X^*$

The affine second fundamental form

- The map

$$\partial^2\phi_K^* = (h_K^*\partial^2 h_K^* + \partial h_K^* \otimes \partial h_K^*) : X \to \operatorname{Sym}^2 X^*$$

  is homogeneous of degree 0
- We assume that $\partial^2\phi_K^*(x)$ is positive definite for each $x \in X\backslash\{0\}$
- It is the affine analogue of the second fundamental form
  - If $X = X^* = \mathbf{R}^n$, $x \in \partial K$, and $v \in T_x\partial K$, then

$$v \cdot \mathrm{II}(x)v = \frac{v \cdot \partial^2\phi_K^*(x)v}{|\partial\phi_K^*(x)|}$$

    for each $x \in \partial K$

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The affine inverse second fundamental form

- The map

$$\partial^2 \phi_K = (h_K \partial^2 h_K + \partial h_K \otimes \partial h_K) : X^* \to \operatorname{Sym}^2 X$$

  is homogeneous of degree 0
- We assume that $\partial^2 \phi_K(\xi)$ is positive definite for each $\xi \in X^* \backslash \{0\}$
- It is the affine analogue of the inverse second fundamental form
  - If $X = X^* = \mathbf{R}^n$, $x \in \partial K$, and $v \in T_x \partial K$, then

$$v \cdot \mathrm{II}^{-1}(x) v = (v \cdot \partial^2 \phi_K(\partial \phi_K^*(x)) v) |\partial \phi_K^*(x)|$$

    for each $x \in \partial K$

The curvature function

- The differential n-form dx on X induces a determinant function $\det : \mathrm{Sym}^2 X \to \mathbf{R}$
- The *curvature function of a convex body* K is defined to be the function $f_K : X^* \to \mathbf{R}$ satisfying

$$\det \partial^2 \phi_K(\xi) = h_K(\xi)^{n+1} f_K(\xi)$$

- The curvature function f_K is homogeneous of degree $-n-1$.
- It is the affine analogue of the reciprocal Gauss curvature
  - If $X = X^* = \mathbf{R}^n$, then

$$\frac{1}{\kappa(x)} = f_K(\nu(x))$$

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The polar curvature function

- The differential n-form $d\xi$ on X^* induces a determinant function $\det : \mathrm{Sym}^2 X^* \to \mathbf{R}$
- The *polar curvature function of a convex body K* is defined to be the function $f_K^* : X \to \mathbf{R}$ satisfying

$$\det \partial^2 \phi_K^*(x) = h_K^*(x)^{n+1} f_K^*(x)$$

- The polar curvature function f_K^* is homogeneous of degree $-n-1$.
- It is the affine analogue of the Gauss curvature
  - If $X = X^* = \mathbf{R}^n$, then

$$\kappa(x) = (x \cdot \nu(x))^{n+1} f_K^*(x),$$

for each $x \in \partial K$

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Valuations

- Let $\mathcal{C}(X)$ denote the space of convex bodies in X
- A valuation is essentially a finitely additive measure on $\mathcal{C}(X)$
- If Y is an additive semigroup and $S \subset \mathcal{C}(X)$, then a *Y-valued valuation on S* is a map $\Phi : S \to Y$ such that

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L),$$

  for each $K, L \in S$ such that $K \cap L, K \cup L \in S$.
- Any integral invariant of a convex body is a valuation

Ingredients for building affine invariant valuations

- Homogeneous contour integral on X or X^*
- Natural homogeneous functions
  - Natural evaluation map $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbf{R}$
  - Identity vector fields $x : X \to X$ and $\xi : X^* \to X^*$
- Homogeneous measures dx on X and $d\xi$ on X^*
- Homogeneous functions associated with a convex body $K \subset X$
  - Support function h_K and polar support function h_K^*
  - Curvature function f_K and polar curvature function f_K^*
  - First and second partial derivatives of support and polar support functions

Proposition

Let $\mathcal{C}^2(X)$ denote the space of all convex bodies in X with C^2 support functions. Given a vector space Y and a measurable function $\psi : X^* \times \mathbf{R} \times X \times \overline{S^2_+ X} \to Y$ such that $\psi(\cdot, h_K(\cdot), \partial h_K(\cdot), \partial^2 h_K(\cdot))$ is homogeneous of degree $-n$, the map $\Psi : \mathcal{C}^2(X) \to Y$ given by

$$\Psi(K) = \oint_{X^*} \psi(\xi, h_K(\xi), \partial h_K(\xi), \partial^2 h_K(\xi))\, d\xi,$$

for each $K \in \mathcal{C}^2(X)$, is a valuation.

The polar projection body of a convex body K

- Given a convex body K and $x \in X$, the area of the shadow in direction x is proportional to

$$h^*_{\Pi^* K}(x) = \frac{1}{V(K)} \oint |\langle \xi, x \rangle| f_K(\xi)\, d\xi.$$

- This defines a new convex body $\Pi^* K \subset X$ naturally associated with K, known as the **polar projection body**.
- The volume of $\Pi^* K$ is given by

$$V(\Pi^* K) = \frac{1}{n} \oint [h^*_{\Pi^* K}(x)]^{-n}\, dx$$

- $V(\Pi^* K)$ can be viewed as an affine average of shadow area and therefore as an affine surface area
- It is equivariant under linear transformations. Given any invertible linear transformation $A : X \to X$ and $x_0 \in X$,

$$\Pi^*(AK) = A\Pi^* K$$

The projection body of a function $f : X \to \mathbf{R}$

Given a smooth decaying function $f : X \to \mathbf{R}$, define the polar projection body $\Pi^* f$ by

$$h^*_{\Pi^* f}(v) = \int |\langle v, \partial f(x) \rangle| \, dx$$

Sharp affine isoperimetric and Sobolev inequalities

Theorem

(Petty projection inequality)

$$V(\Pi^* K) \geq V(K),$$

with equality holding if and only if K is an ellipsoid

Theorem

(Affine Sobolev inequality, G. Zhang, JDG 1999) Given $n > 1$ and $f : X \to \mathbf{R}$, where $\dim X = n$,

$$V(\Pi^* f)^{-1/n} \leq \|f\|_{n/(n-1)}.$$

Equality if and only if f is a generalized Gaussian.

Duality

Given the function $\psi : X^* \times \mathbf{R} \times X \times \overline{S^2_+ X} \to Y$ as before, there exists a dual function $\psi^* : X \times \mathbf{R} \times X^* \times \overline{S^2_+ X^*} \to Y$ such that

$$\begin{aligned} &\oint_{X^*} \psi(\xi, h_K(\xi), \partial h_K(\xi), \partial^2 h_K(\xi))\, d\xi \\ &\qquad = \oint_X \psi^*(x, h_{K^*}(x), \partial h_{K^*}(x), \partial^2 h_{K^*}(x))\, dx \end{aligned}$$

for each $K \in \mathcal{C}^2(X)$.

Homogeneous scalar valuations

- Only homogeneous scalar functions on X^* associated with a convex body K are its support function h_K and curvature function f_K.
- $\det \partial^2 \phi_K = h_K^{n+1} f_K$ is homogeneous of degree 0.
- For each $q \in (-\infty, \infty)$,

$$A_q(K) = \oint_{X^*} (h_K^{n+1} f_K))^{\frac{q+n}{2n}} (h_K(\xi))^{-n} \, d\xi$$

  defines a homogeneous valuation
- If $q \in [-n, n]$, then A_q can be extended to a $\mathrm{GL}(X)$-valuation of degree q on $\mathcal{C}(X)$.
- $A_n(K) = cV(K)$ and $A_{-n}(K) = c^* V(K^*)$
- $A_q(K)$ is the L_p affine surface area, where

$$p = \frac{n(n-q)}{n+q}.$$

Theorem of Ludwig and Reitzner

Any upper semicontinuous real-valued valuation on $\mathcal{C}(X)$ that is $\mathrm{GL}(X)$-homogeneous of degree $q \in [-n, n]$ is, up to a constant factor, equal to A_q.

The Legendre Transform

- Given $K \in \mathcal{C}^2(X)$ such that $K^* \in \mathcal{C}^2(X^*)$, the differential of ϕ_K, $\partial\phi_K : X^* \to X$ is a homogeneous diffeomorphism
- If $x = \partial\phi_K(\xi)$, then

$$\begin{aligned} h_{K^*}(x) &= h_K(\xi) \\ \xi &= \partial\phi_{K^*}(x) \\ \partial^2\phi_K(\xi)\partial^2\phi_{K^*}(x) &= I \end{aligned}$$

Hug's Theorem

Since

$$dx = \det \partial^2 \phi_K(\xi)\, d\xi,$$

it follows that

$$\begin{aligned}
A_q(K^*) &= \oint_{X*} (\det \partial^2 \phi_{K^*}(x))^{\frac{q+n}{2n}} (h_{K^*}(x))^{-n}\, dx \\
&= \oint_{X^*} (\det \partial^2 \phi_K(\xi))^{-\frac{q+n}{2n}} (h_K(\xi))^{-n} \det \partial^2 \phi_K(\xi)\, d\xi \\
&= \oint_{X^*} (\det \partial^2 \phi_K(\xi))^{\frac{-q+n}{2n}} (h_K(\xi))^{-n}\, d\xi \\
&= A_{-q}(K)
\end{aligned}$$

Ellipsoid-valued valuations

- Naturally associated to K are two ellipsoids $E_2 K$ and $E_{-2} K$
- The support function of the Legendre ellipsoid is given by

$$h^2_{E_2 K}(\xi) = \frac{1}{V(K)} \oint \left(\frac{\langle \xi, x \rangle}{h^*_K(x)} \right)^2 h^*_K(x)^{-n}\, dx$$

- The polar support function of the ellipsoid defined by Lutwak, Yang, and Zhang is given by

$$(h^*_{E_{-2} K})^2(v) = \frac{1}{V(K)} \oint \left(\frac{\langle \xi, v \rangle}{h_K(\xi)} \right)^2 h_K(\xi) f_K(\xi)\, d\xi$$

- Ludwig has established that under reasonable assumptions these are the only ellipsoid-valued valuations

The Cramer-Rao inequality for convex bodies

Theorem

(Lutwak-Yang-Zhang) If K is a convex body, then

$$E_{-2}K \subset E_2K$$

with equality holding if and only if K is an ellipsoid centered at the origin.

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Elementary inclusion lemma

Lemma

If $K, L \subset X$ *satisfy*

$$\langle \xi, x \rangle \leq h_K(\xi) h_L^*(x)$$

for each $x \in X$ *and* $\xi \in X^*$, *then* $L \subset K$.

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Proof of the Cramer-Rao inequality for convex bodies

Observe that

$$\langle \xi, v \rangle = [\langle v, \partial \rangle, \langle \xi, x \rangle]$$

For each $\xi \in X^*$ and $v \in X$,

$$\begin{aligned}
\langle \xi, v \rangle &= \frac{1}{nV(K)} \oint_X [\langle v, \partial \rangle, \langle \xi, x \rangle](h_K^*)^{-n}(x)\, dx \\
&= -\frac{1}{V(K)} \oint_X \langle \xi, x \rangle \langle v, \partial h_K^*(x) \rangle (h_K^*)^{-n-1}(x)\, dx \\
&\leq \left(\frac{1}{V(K)} \oint_X \langle \xi, x \rangle^2 (h_K^*)^{-n-2}(x)\, dx \right)^{1/2} \\
&\quad \cdot \left(\frac{1}{V(K)} \oint_X \langle v, \partial h_K^*(x) \rangle^2 (h_K^*)^{-n}(x)\, dx \right)^{1/2} \\
&= h_{E_2 K}(\xi) h^*_{E_{-2} K}(v)
\end{aligned}$$

Other valuations

- There are similar constructions using the homogeneous contour integral of $\mathrm{GL}(X)$-homogeneous vector-, tensor-, and body-valued valuations
  - Center of mass
  - Legendre and LYZ ellipsoids
  - L_p-centroid bodies
  - L_p-projection bodies
- Theorems of Ludwig show that these constructions produce all possible continuous $\mathrm{GL}(X)$-homogeneous valuations
- These also satisfy sharp affine geometric inequalities, where equality holds if and only if it is an ellipsoid

Sharp affine Sobolev inequalities

- There are corresponding invariants associated to a smooth function (or probability density) on X.
- These satisfy sharp affine Sobolev inequalities (which imply the classical sharp Sobolev inequalities of Aubin and Talenti:
- Equivalent to information theoretic inequalities for the entropy, p-th moment, and generalized Fisher information of a probability distribution
- Equality holds if and only if distribution is a generalized Gaussian
- Even the 1-dimensional case is interesting:
  E. Lutwak, D. Yang, G. Zhang. *Cramer-Rao and moment-entropy inequalities for Renyi entropy and generalized Fisher information*, **IEEE Transactions on Information Theory 51** (2005) 473-478.