CHENG AND YAU'S WORK ON THE MONGE-AMPÈRE EQUATION AND AFFINE GEOMETRY

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1. Introduction

S. T. Yau has done extremely deep and powerful work in differential geometry and partial differential equations. His resolution of the Calabi conjecture on the existence of Kähler-Einstein metrics, by solving a complex Monge-Ampère equation on Kähler manifolds, is of fundamental importance in both mathematics and physics.

We would like to recall in this article the contributions of S. Y. Cheng and S. T. Yau to the real Monge-Ampère equation and its applications to affine geometry. Many definitions and details are omitted here. We refer the reader to the surveys by Loftin [34] and Trudinger and Wang [51], as well as other references cited below, for more extensive discussions of the topics covered here.

2. The Monge-Ampère equation

The Monge-Ampère equation is a fully nonlinear PDE, where the highest order term is the determinant of the Hessian of the function to be solved for. Since it was introduced by Monge and Ampère about two hundred years ago, it has been studied extensively by mathematicians, including Minkowski, Lewy, Aleksandrov before the 1950's and many more since then. During the last century the main motivation for studying the Monge-Ampère equation has been geometric applications such as the Minkowski problem, the existence of local isometric embeddings of a 2-dimensional Riemannian manifolds in $\mathbb{R}^3$, the corresponding global question, known as the Weyl problem, for a positively curved closed 2-dimensional Riemannian manifold, and the classification of affine spheres.

The Monge-Ampère equation has very nice properties. It is invariant under unimodular linear transformations, and it prescribes the Jacobian of the gradient mapping. It arises naturally when prescribing the Gauss curvature of a hypersurface in Euclidean space. The existence and uniqueness of generalized convex solutions to Monge-Ampère equations were obtained by Minkowski, Lewy, and Aleksandrov. The regularity of generalized solutions, however, was an extremely difficult problem, due to the strong nonlinearity of the equation.

The first results on regularity were in dimension two. Regularity of weak convex solutions follows from Morrey's regularity results for two dimensional uniformly elliptic equations. The existence of smooth solutions to the Minkowski problem and the Weyl problem was established indepndently by Nirenberg and Pogorelov in early 1950s. Heinz established a sharp $C^{2,\alpha}$ estimate for the two dimensional Monge-Ampère equation by using the partial Legendre transform to convert the Monge-Ampère equation into a quasilinear elliptic PDE.

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Establishing regularity in higher dimensions is much more challenging. Shing-Tung Yau, in collaboration with Shiu-Yuen Cheng, made substantial contributions in the 1970s. They used *a priori* estimates of Calabi and Pogorelov to establish regularity theorems for the Minkowski problem [11] and the Dirichlet problem for the Monge-Ampère equation [12]. They also resolved a conjecture of Calabi on hyperbolic affine spheres [12, 14]. We describe this work below.

3. Cheng and Yau's work on the Dirichlet problem

Throughout this section, given a function u on a domain of $\mathbb{R}^n$, we denote its gradient by Du and its Hessian by D^2u.

The classical Dirichlet problem for the Monge-Ampère equation asks the following: Given a bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ and continuous functions $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to (0, \infty)$ and $\phi : \partial\Omega \to \mathbb{R}$, does there exist a unique function u satisfying

$$\begin{aligned} \det D^2 u &= f(\cdot, u, Du), \text{ in } \Omega, \\ u &= \phi, \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Aleksandrov [1] (and Bakelman [2] for dimension two) established the existence and uniqueness of generalized solutions Cheng and Yau [12] established the existence of smooth solutions to the Dirichlet problem where Ω is a smooth, uniformly convex domain in $\mathbb{R}^n$ and ϕ is a C^2 function.

Calabi [7] and Pogorelov [43, 42, 45] established interior a priori C^2 and C^3 bounds for strictly convex solutions to Monge-Ampère equations, but it was not known how to prove that the solution is strictly convex. Cheng and Yau accomplished this by first studying solutions with f singular near the boundary $\partial\Omega$, such that $|Du(x)| \to \infty$ as $x \to \partial\Omega$. In that case, the Legendre transform of u, which we denote by u^*, is a convex, uniformly Lipschitz continuous function defined on the entire space $\mathbb{R}^n$ and satisfies the Monge-Ampère equation $\det D^2 u^* = 1/f$. The *a priori* estimates of Calabi and Pogorelov can then be used to show that u is strictly convex.

To establish the existence of smooth solutions, Cheng and Yau reduce the question to the Minkowski problem and apply Aleksandrov's theorems on the uniqueness of generalized solutions to the Dirichlet problem and the Minkowski problem, as well as their own theorem on the existence of smooth solutions to the Minkowski problem (see section 9).

As was pointed out by Cheng and Yau, some of their results on the Dirichlet problems and affine spheres were obtained independently by Calabi and Nirenberg. Pogorelov announced a proof in early 1970s, but details of his proof were not published much later in his book [45].

4. Subsequent work on the Monge-Ampère equation

The Monge-Ampère equation has become an active area of study since the work of Cheng and Yau. We describe briefly some of the highlights related to their work on the Dirichlet problem.

The global regularity of solutions to the Dirichlet problem (1) was later obtained independently by Caffarelli-Nirenberg-Spruck [4] and by Krylov [29] in early 1980s, assuming that $\partial\Omega \in C^{3,1}$ and $\phi \in C^{3,1}$. Caffarelli also proved the strict convexity of solutions to the Monge-Ampère equation when f satisfies a doubling condition [6]; the interior $C^{2,\alpha}$ estimate when $f \in C^\alpha$; and the interior $W^{2,p}$ estimate when $f \in C^0$ [5]. The global $C^{2,\alpha}$ estimate under the sharp conditions $\partial\Omega \in C^3$ and $\phi \in C^3$ was obtained by Trudinger and Wang [50]. Wang also found an example that showed the continuity of f is necessary for the $W^{2,p}$ estimate for large p [51]. These estimates have been used to establish the regularity for the affine maximal surface equation [51]. In recent years, there has also been renewed interest in Monge-Ampère type equation due to applications in the optimal transportation.

Also related to the Dirichlet problem is the existence of a hypersurface with constant Gauss curvature and prescribed boundary in Euclidean space, which was studied and solved by Guan and Spruck [22] and Trudinger and Wang [49]. Urbas [52] studied the boundary regularity of a hypersurface with prescribed Gauss curvature in a domain.

5. Affine Spheres

As part of their great burst of activity in the late 1970s, Cheng and Yau proved many geometric results concerning differential structures invariant under affine transformations of $\mathbb{R}^n$. We describe this work here. A more extensive discussion on the topics discussed in this and the following two sections can be found in the survey by Loftin [34].

Affine differential geometry is the study of those differential properties of hypersurfaces in $\mathbb{R}^{n+1}$ which are invariant under volume-preserving affine transformations. One way to develop this theory is to start with the *affine normal*, which is an affine-invariant transverse vector field to a convex C^3 hypersurface. A hypersurface is an *affine sphere* if the lines formed by the affine normals all meet at a point, called the center. A convex affine sphere is called *elliptic*, *parabolic*, or *hyperbolic* according to whether the affine normals point toward the center, are parallel (the center being at infinity), or away from the center, respectively.

The global theory of elliptic and parabolic affine spheres is quite tame: Every properly immersed elliptic affine sphere is an ellipsoid, while every properly immersed parabolic affine sphere is a paraboloid. In this generality, both these results follow from Cheng-Yau's paper [14], in which they show that any properly immersed affine sphere must have complete affine metric. Then one may appeal to earlier results of Calabi [9] to classify global elliptic and parabolic affine spheres. The global classification of parabolic affine spheres is an extension of the well-known Bernstein theorem for the Monge-Ampère equation, which was proved by Jörgens [26] for $n = 2$, Calabi [7] for $n \leq 5$, and Pogorelov [44, 45] for all $n \geq 2$.

Calabi realized that hyperbolic affine spheres are more varied, by noting that two quite different convex cones contain hyperbolic affine spheres asymptotic to their boundaries. In addition to the hyperboloid, which is asymptotic to a round cone over an ellipsoid, Calabi also wrote down an affine sphere asymptotic to the boundary of the first orthant in $\mathbb{R}^{n+1}$, which is a cone over an n-dimensional simplex [9]. Based on these explicit examples in these two extremal cases of convex cones, Calabi conjectured that each proper convex cone admits a unique (up to scaling) hyperbolic affine sphere, and that every properly immersed hyperbolic affine sphere in $\mathbb{R}^{n+1}$ is asymptotic to the boundary of such a cone. Moreover, he conjectured that the proper immersion of a hyperbolic affine sphere is equivalent to the completeness of an intrinsic affine (or Blaschke) metric.

Cheng-Yau [14, 12] prove Calabi's conjecture on hyperbolic affine spheres. Calabi-Nirenberg proved the same results as in [14] around the same time in unpublished work. One of the main techniques in Cheng-Yau's proof is a gradient estimate on a height function. (There are also clarifications of Cheng-Yau's proof in Li [31, 32].)

6. Hyperbolic Affine Spheres and Real Monge-Ampère Equations

If $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$ is a convex domain, then the existence of a hyperbolic affine sphere asymptotic to the cone over Ω follows from the solution of the following Dirichlet problem for a real Monge-Ampère equation

$$\det u_{ij} = \left(-\frac{1}{u}\right)^{n+2}, \qquad (u_{ij}) > 0, \qquad u\big|_{\partial\Omega} = 0. \tag{2}$$

Calabi conjectured that there is a unique solution to (2) on any convex bounded Ω [9]. Cheng-Yau show there always exists such a solution in [12], and uniqueness follows easily by the maximum principle. From this solution, one may use a duality result of Calabi (known to experts at the time of Cheng-Yau's work and published later in [20]), or a later argument of Sasaki [46], to produce the hyperbolic affine sphere asymptotic to the cone over Ω.

As noted above, the paper [12] was one of the first works to prove the existence smooth solutions to general real Monge-Ampère equations on convex domains. The technique is to prove regularity of Alexandrov's weak solution. A key step in the proof is to approximate solutions to the Dirichlet problem for a real Monge-Ampère equation by solutions to the Minkowski problem on $\mathbb{S}^n$, which are provided by Cheng-Yau in [11].

Loewner-Nirenberg solved (2) earlier in the case of domains in $\mathbb{R}^2$ with smooth boundary [33]. Cheng-Yau's result requires no regularity of $\partial\Omega$ except that provided by convexity. The solution of (2) for $\partial\Omega$ only Lipschitz relies on using Calabi's explicit solution on a simplex as a barrier. The case of rough boundary is of particular geometric interest, as most convex domains admitting cocompact projective group actions have boundaries which are nowhere C^2 [27, 3].

Cheng-Yau provide another existence and regularity proof for the real Monge-Ampère equation in [13], this time using a tube domain construction to gain access to Yau's estimates for the complex Monge-Ampère equation [55].

7. Affine Manifolds

An affine manifold is a manifold with coordinate charts in $\mathbb{R}^n$ and affine gluing maps $x \mapsto Ax + b$. The tangent bundle to an affine manifold carries a natural complex structure, which can be provided by gluing together tube domains over the affine coordinate charts in $\mathbb{R}^n$. There is also a natural notion of an affine Kähler, or Hessian, metric, which is the natural restriction of a Kähler metric on the total space of the tangent bundle which is invariant in the bundle directions.

In [13], Cheng-Yau asked and answered the analogous question for Yau's solution to the Calabi conjecture on Kähler manifolds—the existence of Kähler-Einstein metrics. In the Kähler case, any closed Kähler manifold with first Chern class $c_1 < 0$ admits a unique Kähler-Einstein metric of negative Ricci curvature. Cheng-Yau define an affine first Chern class and prove the negativity of this affine Chern class on a closed affine Kähler manifold is equivalent to the existence of a complete Kähler-Einstein metric on the tangent bundle. The restriction of this metric to the affine manifold is nowadays called the Cheng-Yau metric. The class of affine manifolds admitting a complete Cheng-Yau metric is exactly the class of affine quotients of proper convex cones. Moreover, the Cheng-Yau metric on a convex cone and the hyperbolic affine sphere asymptotic to the cone are equivalent [47, 35]. (Note the Cheng-Yau metric is not in general Einstein: it is merely the restriction of the Kähler-Einstein metric on the tangent bundle.)

The affine analog of a Calabi-Yau manifold (a closed Kähler manifold with $c_1 = 0$) is an affine Kähler manifold admitting a parallel volume form. On any closed affine Kähler manifold with parallel volume, Cheng-Yau construct a flat affine-Kähler metric by appealing to estimates in [55].

8. Maximal Hypersurfaces in Minkowski Space

Calabi conjectured in [8] that every entire maximal spacelike hypersurface in Minkowski space $\mathbb{R}^{n,1}$ must be a hyperplane, and provided a proof for $n \leq 4$. In [10], Cheng-Yau prove this result for all n, using estimates similar to those in [14]. This Bernstein property for maximal spacelike hypersurfaces is in contrast to the case of entire minimal graphs in $\mathbb{R}^{n+1}$. These must be linear for $n < 8$, but there are entire nonlinear examples for $n \geq 8$.

9. The Minkowski problem

The next few sections describe the background to the Minkowski problem, which is a question about the affine geometry of convex bodies in $\mathbb{R}^n$, i.e. convex bodies in an abstract finite-dimensional real vector space. However, it is often convenient to use geometric aspects special to $\mathbb{R}^n$ such as the standard unit sphere. We therefore will use the following notation and definitions:

- Denote $X = \mathbb{R}^n$ and let X^* be the dual vector space to X (which is also $\mathbb{R}^n$).
- Denote the natural contraction between $x \in X$ and $\xi \in X^*$ by $\langle \xi, x \rangle$ or $\langle x, \xi \rangle$.
- Denote the standard unit sphere in X or X^* by S^{n-1}.
- The group of invertible linear transformations of X will be denoted by $\mathrm{GL}(n)$ and identified with invertible square matrices.
- Any $A \in \mathrm{GL}(n)$ induces a transpose linear transformation, which we denote by $A^t : X^* \to X^*$.
- The inverse of A^t will be denoted $A^{-t} : X^* \to X^*$.

10. Convex geometry without smoothness assumptions

Everything in this section is stated without proof. Details can be found in standard references such as [18, 19, 48].

A set $K \subset X$ is called a *convex body,* if it is compact and convex and has non-empty interior.

The volume of K with respect to the standard Lebesgue measure on $X = \mathbb{R}^n$ will be denoted $V(K)$.

10.1. **Support function.**

The support function of a convex body is the most fundamental and convenient way to represent a convex body. If the body is a polytope, then the support function is given by a finite set of data. In general, the support function is a convex homogeneous function, which can therefore be restricted to the unit sphere without any loss of information.

The *support function* $h_K : X^* \to \mathbb{R}$ of a convex body K is defined to be

$$h_K(u) = \sup_{x \in K} \langle u, x \rangle,$$

for each $u \in X^*$. Observe that since the function h_K is given by the supremum of linear functions, it is a convex function homogeneous of degree 1. Moreover, the body K can be reconstructed from h_K by

$$K = \{x \; : \; \langle u, x\rangle \leq h_K(u), \text{ for each } u \in X^*\}.$$

10.2. **Invariance properties of the support function.** If $\widetilde{K} = AK + v$, where $A \in \mathrm{GL}(n)$ and $v \in X$, then

$$h_{\widetilde{K}}(u) = h_K(Au) + \langle u, v\rangle,$$

for each $u \in X^*$.

10.3. **Minkowski sum.** If K and L are convex bodies in $\mathbb{R}^n$, then so is its Minkowski sum $K + L$, which is given by

$$K + L = \{x + y \; : \; x \in K, \; y \in L\}.$$

The sum can also be defined using support functions by

$$h_{K+L} = h_K + h_L.$$

10.4. **Mixed volume.** Given a compact set $S \subset \mathbb{R}^n$, let $V(S)$ denote the volume of S with respect to standard Lebesgue measure on $\mathbb{R}^n$.

Let B denote the standard unit ball in $\mathbb{R}^n$. If K is a convex body, then the limit

$$V(K, B) = \lim_{t \to 0+} \frac{V(K + tB) - V(K)}{t}$$

exists and that if K has smooth boundary ∂K, then $V(K, B)$ is the surface area (i.e., $(n-1)$-dimensional Hausdorff measure) of ∂K.

This leads naturally to the following generalized surface area of a convex body K with respect to another convex body L:

$$V(K, L) = \lim_{t \to 0+} \frac{V(K + tL) - V(K)}{t}.$$

This is often called the *mixed volume* of K and L.

10.5. **Surface area measure.** The mixed volume $V(K, L)$ is a bounded linear functional of the support function $h_L : S^{n-1} \to \mathbb{R}$. Using the Riesz representation theorem, there exists a unique measure S_K on the unit sphere S^{n-1} such that

$$V(K, L) = \int_{S^{n-1}} h_L(u)\, dS_K(u),$$

for each convex body L. The measure S_K is called the *surface area measure* of K. Note that this is related to but *not* what differential geometers would call the surface area measure of the boundary ∂K, but it turns out to be closely related. Also, see Yang [54] for a different approach to defining the surface area measure that avoids integration over the unit sphere (which obscures the behavior of the formula under affine transformations).

10.6. **Invariance properties of the surface area measure.** Since the measure S_K is defined on the unit sphere, and the unit sphere is not preserved under affine transformations, the behavior of the measure S_K under affine transformations is quite complicated. On the other hand, the associated integral

$$h \mapsto \int_{S^{n-1}} h(u)\, dS_K(u)$$

behaves quite nicely, since it is the essentially the same as the functional $L \mapsto V(K, L)$.

In particular, if $A \in \mathrm{GL}(n)$ and $v \in X$, then since $V(AK + v) = |\det A|V(K)$,

$$V(AK + v, L) = |\det A|V(K, A^{-1}L).$$

It follows that if $\widetilde{K} = AK + v$, then

$$\int_{S^{n-1}} h(u)\, dS_{\widetilde{K}}(u) = |\det A| \int_{S^{n-1}} h(A^{-t}u)\, dS_K(u).$$

Also, observe that since $V(K, L + v) = V(K, L)$ for each $v \in X$, it follows that

$$\int_{S^{n-1}} \langle u, v\rangle\, dS_K(u) = 0, \tag{3}$$

for each $v \in X$.

10.7. **The Minkowski problem.** The Minkowski problem asks the following: Given a measure μ on the unit sphere $S^{n-1} \subset X^*$ such that

$$\int_{S^{n-1}} u\, d\mu(u) = 0, \tag{4}$$

does there exist a unique convex body K whose surface area measure S_K satisfies $S_K = \mu$?

Solutions to the Minkowski problem were given by H. Minkowski, A. D. Aleksandrov, and Fenchel and Jessen (see the book by Schneider [48] for references).

10.8. **The Brunn-Minkowski inequality.** The Brunn-Minkowski inequality for convex bodies states that

$$V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},$$

with equality holding if and only if $L = tK + v$ for some $t \in [0, \infty)$ and $v \in X$. It immediately implies the generalized isoperimetric inequality

$$V(K, L) \geq V(K)^{(n-1)/n} V(L)^{1/n}, \tag{5}$$

with the same equality condition as above. If L is the standard unit ball, (5) is the classical Euclidean isoperimetric inequality for a convex body.

10.9. **Uniqueness in the Minkowski problem.** If K and L are both solutions to the Minkowski problem, then

$$\begin{aligned} V(L) &= V(L, L) \\ &= \int_{S^{n-1}} h_L(u)\, dS_L(u) \\ &= \int_{S^{n-1}} h_L(u)\, d\mu(u) \\ &= \int_{S^{n-1}} h_L(u)\, dS_K(u) \\ &= V(K, L) \\ &\geq V(K)^{(n-1)/n} V(L)^{1/n}, \end{aligned}$$

which implies that

$$V(L) \geq V(K).$$

All of the above also holds, if K and L are swapped. Therefore, equality holds throughout. By this and the equality condition of the Minkowski inequality, it follows that L is a translate of K.

10.10. **Variational approach to the Minkowski problem.** The inequality (5) can be written as

$$V(L)^{-1/n} \int_{S^{n-1}} h_L(u)\, dS_K(u) \geq V(K)^{(n-1)/n},$$

with equality holding if and only if $L = tK + v$ for some $t \in [0, \infty)$ and $v \in X$. Therefore, one possible approach to solving the Minkowski problem is to minimize the functional

$$L \mapsto \int_{S^{n-1}} h_L(u)\, d\mu(u)$$

over all convex bodies L with volume equal to 1.

This approach can be used to solve the Minkowski problem when μ is a discrete measure and K is a convex polytope. An approximation argument extends the solution to arbitrary convex bodies.

11. Convex geometry with smoothness assumptions

No proofs are provided for any of the statements in this section, but everything can be established using straightforward calculations and therefore are left as an elementary exercise for the reader.

If the boundary of the convex body is assumed to be C^2 and have positive definite second fundamental form, then the concepts above can be described in terms of better known local differential geometric invariants such as the Gauss map and second fundamental form of the boundary.

First, observe that if the support function h_K is C^2, then, since it is convex, its Hessian $\partial^2 h_K(u)$ is always positive semi-definite. Moreover, since h_K is homogeneous of degree 1, its differential ∂h_K is homogeneous of degree 0. Therefore, its Hessian $\partial^2 h_K(u)$ always has at least one zero eigenvalue corresponding to the eigenvector u.

Throughout this section, we assume that $K \subset X$ is a convex body such that its support function $h_K : X^* \to \mathbb{R}$ is C^2 and its Hessian $\partial^2 h_K(u)$ has rank $n-1$ for each $u \neq 0$. Using the inverse Gauss map defined below and the inverse function theorem, it follows that these assumptions are equivalent to boundary ∂K being a C^2 hypersurface with a positive definite second fundamental form everywhere.

11.1. **The inverse Gauss map.** Under the assumptions above, the differential $\partial h_K : X^* \to X$ is homogeneous of degree 0, and its image is the boundary ∂K. The restriction of ∂h_K to the unit sphere S^{n-1} is the inverse to the Gauss map $\gamma_K : \partial K \to S^{n-1}$, where $\gamma_K(x)$ is the outer unit normal to ∂K at $x \in \partial K$.

11.2. **The inverse second fundamental form.** Recall that the second fundamental form of ∂K at $x \in \partial K$ is defined to be the differential of the Gauss map at x. Therefore, since ∂h_K restricted to the unit sphere is the inverse map to the Gauss map γ_K, it follows that, if $u \in S^{n-1}$, then the Hessian $\partial^2 h_K(u)$ restricted to the hyperplane $u^\perp$ is the inverse of the second fundamental form at $x = \partial h_K(u)$.

The positive eigenvalues of $\partial^2 h_K(u)$ are called the *radii of curvature* of ∂K at $x = \partial h_K(u)$. Their reciprocals are the eigenvalues of the second fundamental form at x and are called the *principal curvatures*.

11.3. **The curvature function.** In affine convex geometry, we want to define the curvature function $f_K : X^*\backslash\{0\} \to (0, \infty)$ of the convex body K to be the determinant of the positive definite $(n-1)$-by-$(n-1)$ minor of $\partial^2 h_K$ (which has rank $n-1$). However, there is no affine invariant way of isolating this minor and taking its determinant directly. Instead, we define it indirectly.

Define the *curvature function* $f_K : X^*\backslash\{0\} \to (0, \infty)$ to be the homogeneous function of degree $-n-1$ given by

$$f_K = h_K^{-n-1} \det \partial^2 \left(\frac{1}{2} h_K^2 \right). \tag{6}$$

That this is the right definition of the curvature function can be seen as follows. If $u \in S^{n-1}$, then

$$f_K(u) = \det \partial^2 h_K(u)\big|_{u^\perp}.$$

On the other hand, recall that the Gauss curvature $\kappa(x)$ is the determinant of the second fundamental form at $x \in \partial K$. Since $\partial^2 h_K(u)\big|_{u^\perp}$ and the second fundamental form of ∂K at $x = \partial h_K(u)$ are inverses of each other, it follows that $f_K(u)$ is the reciprocal of the Gauss curvature at x.

11.4. **The surface area measure.** The surface area measure is given by $dS_K(u) = f_K(u)\, du$, where du is the standard $(n-1)$-dimensional volume measure on the unit sphere S^{n-1}.

By (3), the curvature function f_K satisfies the identity

$$\int_{S^{n-1}} u f_K(u)\, du = 0.$$

11.5. **The Minkowski problem.** If the measure μ can be written as $d\mu(u) = \phi(u)\,du$, for a positive continuous function ϕ, then the Minkowski problem can be restated as follows:

If ϕ is a positive continuous function on the unit sphere S^{n-1} satisfying

$$\int_{S^{n-1}} u\phi(u)\,du = 0,$$

find the unique convex body K whose curvature function f_K is equal to ϕ.

Another equivalent restatement is:

If ψ is a positive continuous function on the unit sphere S^{n-1} satisfying

$$\int_{S^{n-1}} \frac{u}{\psi(u)}\,du = 0,$$

find the unique convex body K whose Gauss map $\gamma_K : \partial K \to S^{n-1}$ and Gauss curvature $\kappa : \partial K \to (0,\infty)$ satisfy for each $x \in \partial K$,

$$\kappa(x) = \psi(\gamma_K(x)).$$

11.6. **The Minkowski problem as a PDE.** The Minkowski problem can be described roughly as prescribing the determinant of the Hessian of the support function and therefore a PDE of Monge-Ampère type. This is not a precise description, because the Hessian is in fact singular and its determinant is zero.

As mentioned in Cheng-Yau [11], there are two different ways to write the Minkowski problem rigorously as a PDE of Monge-Ampère type.

First, if the support and curvature functions are restricted to the unit sphere, then they satisfy the equation

$$\det(\nabla^2 h_K + h_K g) = f_K,$$

where g is the standard Riemannian metric on the unit sphere, $\nabla^2 h_K$ is the Hessian of h_K with respect to this metric, and the determinant is taken with respect to an orthonormal frame. Therefore, if we restrict to convex bodies with C^2 support functions, the Minkowski problem is equivalent to the following PDE of Monge-Ampère type on the sphere:

Find the unique function $h : S^{n-1} \to (0,\infty)$ with positive definite Hessian and satisfying

$$\det(\nabla^2 h + hg) = \phi, \tag{7}$$

where ϕ a continuous positive function on S^{n-1} such that

$$\int_{S^{n-1}} u\phi(u)\,du = 0. \tag{8}$$

Second, if the support and curvature functions are restricted to an affine hyperplane $H \subset \mathbb{R}^n$, then they satisfy the equation

$$\det \partial^2 h_K = f_K;$$

here, $\partial^2 h_K$ denotes usual flat Hessian of h_K restricted to the hyperplane H. For example, if $H = \{x^n = 1\}$, then h_K depends only on the co-ordinates $x^1, \ldots, x^{n-1}$, and here $\partial^2 h_K$ denotes the standard Hessian with respect to those co-ordinates. Therefore, a solution to the Minkowski problem implies a solution h to the classical Monge-Ampère equation on R^{n-1}

$$\det \partial^2 h = \phi,$$

where ϕ is given.

It is worth noting that the definition (6) gives a third way to represent the Minkowski problem as a PDE of Monge-Ampère type:

Given a positive function $\phi : X^*\backslash\{0\}$ homogeneous of degree $-n-1$ such that

$$\int_{S^{n-1}} u\phi(u)\,du = 0,$$

find the unique convex function $h : X^* \to [0,\infty)$ homogeneous of degree 1 such that

$$\det \partial^2 \left(\frac{1}{2}h^2\right) = h^{n+1}\phi.$$

12. Cheng and Yau's regularity theorem for the Minkowski problem

12.1. **Statement.** The paper of Cheng and Yau [11] addresses the regularity of a solution to the Minkowski problem. In other words, do regularity assumptions on curvature function f_K of a convex body K imply corresponding regularity conclusions on the support function h_K? Toward this end, they used the continuity method (see below) to prove the following:

Theorem 1. If $k \geq 3$ and K is a convex body whose curvature function f_K is in $C^k(X^*\backslash\{0\})$, then its support function h_K is in $C^{k+1,\alpha}(X^*\backslash\{0\})$ for all $\alpha \in (0,1)$.

Regularity theorems when $\dim X = 2$ were first obtained by Pogorelov [41] and Nirenberg [39]. Pogorelov [45] proved the theorem above independently.

Regularity theorems of Evans [17] and Krylov [28, 30] can be used to extend the theorem above to all $k \geq 2$. See the survey by Trudinger and Wang [51] for details.

Chou and Wang [15] have given a different proof of regularity using a geometric heat flow, namely the logarithmic Gauss curvature flow.

12.2. **Sketch of Proof.** Cheng and Yau prove regularity by establishing the existence of a smooth solution to the PDE (7), if the given curvature function $\phi : S^{n-1} \to (0,\infty)$ is sufficiently smooth. They do this via the continuity method.

In particular, fix $k \geq 3$ and let $\phi \in C^k(S^{n-1})$ be positive and satisfy (8). Define, for each $t \in [0,1]$, $\phi_t = (1-t)1 + t\phi$ and note that ϕ_t is positive and satisfies (8). Let $S \subset [0,1]$ be the set of all t such that there exists a convex body K_t with support function $h_t \in C^{k+1,\alpha}(S^{n-1})$, for every $\alpha \in (0,1)$, and curvature function equal to ϕ_t. Obviously, $0 \in S$. Therefore, if S is both open and closed, then $1 \in S$ and therefore a solution to (7) exists for the given function ϕ.

That S is open follows by the implicit function theorem and the application of standard linear elliptic PDE theory to the linearization of (7). In particular, the linearized operator is an elliptic self-adjoint operator whose kernel consists exactly of linear functions on X^* restricted to the unit sphere. It therefore can be inverted, and Schauder theory gives the desired regularity for solutions to the linearized equation.

To prove S is closed, Cheng and Yau show how to establish *a priori* C^0, C^1, C^2, and C^3 bounds for a solution to (7). The C^0 bound is equivalent to a diameter bound for the convex body K in terms of its curvature function. This is established using a convex geometric argument.

To obtain the higher regularity bounds, it is first necessary to obtain a lower bound on the diameter of the convex body K. This again is established using a convex geometric argument.

The remaining bounds are now obtained using the second formulation of the Minkowski problem given in section 11.6 on an affine hyperplane. Cheng and Yau show that in this setting the lower bound on the diameter of K allows one to control the size of a level set for the support function h. This allows them to apply *a priori* second and third derivative estimates obtained by Pogorelov [43, 42, 45] for a solution to the Monge-Ampère equation on a bounded convex domain.

Pogorelov's proof of the third derivative estimate is based on earlier work of Calabi [7] and is much more involved than the second derivative bound. Regularity theorems of Evans [17] and Krylov [28, 30] imply $C^{2,\alpha}$ bounds without requiring the C^3 bound.

Higher regularity of the support function now follows by the standard bootstrapping of Schauder estimates using the elliptic PDE (7).

13. Generalizations of the Minkowski problem

The impact of the work of Cheng and Yau [11, 12] on the Minkowski problem goes well beyond the theorems proved in their papers. Their papers are among the first to combine techniques from both the Brunn-Minkowski theory in convex geometry and nonlinear elliptic PDE theory. Until their work, the two worlds of convex geometry and PDE's were largely separate. Their work created a solid bridge between the two subjects, one that has become well traveled by people from both specialties.

This, for example, can be seen by recent work on the L_p Minkowski problem, which was introduced by Lutwak [36, 37]. Lutwak showed that all of the concepts defined in section 10, which he called the L_1 Brunn-Minkowski theory, can be extended to an L_p Brunn-Minkowski theory. In particular, he defined the L_p curvature function $f_p = h^{1-p} f$ of a convex body K, where h is the support function and f the curvature function of K, and formulated the L_p Minkowski problem:

Given a function $\phi : X^* \to (0, \infty)$ homogeneous of degree $-n-p$, find the unique convex body $K \subset X$ whose L_p curvature function is given by ϕ.

This has been studied extensively by both people in PDE's and those in convex geometry. See, for example, the papers of Lutwak, Yang, and Zhang [38], Chou and Wang [16] and Hug, Lutwak, Yang, and Zhang [25], as well as the references contained within them. More recently, Haberl, Lutwak, Yang, and Zhang [24] have formulated and studied the even more general Orlicz-Minkowski problem.

Other Minkowski-type problems have also been studied by Oliker [40], Wang [53], Guan and Li [23].

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