A NEW ELLIPSOID ASSOCIATED WITH CONVEX BODIES

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Abstract. It is shown that corresponding to each convex body there is an ellipsoid that is in a sense dual to the Legendre ellipsoid of classical mechanics. Sharp affine isoperimetric inequalities are obtained between the volume of the convex body and that of its corresponding new ellipsoid. These inequalities provide exact bounds for the isotropic constant associated with the new ellipsoid. Among other things, this leads to a new approach to establishing Ball's maximal shadows conjecture (for symmetric convex bodies).

Corresponding to each origin–symmetric convex (or more general) subset of Euclidean n-space, $\mathbb{R}^n$, there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the convex set is the same about *every* 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the Legendre ellipsoid of the convex set. The Legendre ellipsoid and its polar (the Binet ellipsoid) are well-known concepts from classical mechanics. See Milman and Pajor [MPa1, MPa2], Lindenstrauss and Milman [LiM] and Leichtweiß [Le] for some historical references.

It has slowly come to be recognized that along side the Brunn-Minkowski theory there is a dual theory. The nature of the duality between this dual Brunn-Minkowski theory and the Brunn-Minkowski theory is subtle and not yet understood. It is easily seen that the Legendre (and Binet) ellipsoid is an object of this dual Brunn-Minkowski theory. This observation leads immediately to the natural question regarding the possible existence of a dual analog of the classical Legendre ellipsoid in the Brunn-Minkowski theory. It is the aim of this paper to demonstrate the existence of precisely this dual object. In retrospect, one may well wonder why

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the new ellipsoid presented in this note was not discovered long ago. The simple answer is that the definition of the new ellipsoid becomes obvious only with the notion of L_2-curvature in hand. However, the Brunn-Minkowski theory was only recently extended to incorporate the new notion of L_p-curvature (see [L2], [L3]).

A positive definite $n \times n$ real symmetric matrix A generates an ellipsoid, $\epsilon(A)$, in $\mathbb{R}^n$, defined by

$$\epsilon(A) = \{x \in \mathbb{R}^n : x \cdot Ax \le 1\},$$

where $x \cdot Ax$ denotes the standard inner product of x and Ax in $\mathbb{R}^n$.

Associated with a star-shaped (about the origin) set $K \subset \mathbb{R}^n$ is its Legedre ellipsoid, $\Gamma_2 K$, which is generated by the matrix $[m_{ij}(K)]^{-1}$ where

$$m_{ij}(K) = \frac{n+2}{V(K)} \int_K (e_i \cdot x)(e_j \cdot x)\, dx,$$

with $e_1, \dots, e_n$ denoting the standard basis for $\mathbb{R}^n$ and $V(K)$ denoting the n-dimensional volume of K.

We will associate a new ellipsoid $\Gamma_{-2} K$ with each convex body $K \subset \mathbb{R}^n$. One approach to defining $\Gamma_{-2} K$ without introducing new notation is to first define it for polytopes and then use approximation (with respect to the Hausdorff metric) to extend the definition to all convex bodies.

Suppose $P \subset \mathbb{R}^n$ is a polytope that contains the origin in its interior. Let $u_1, \dots, u_N$ denote the outer unit normals to the faces of P, let $a_1, \dots, a_N$ denote the areas (i.e., $(n-1)$-dimensional volumes) of the corresponding faces and let $h_1, \dots, h_N$ denote the distances from the origin to the corresponding faces. The ellipsoid $\Gamma_{-2} P$ is generated by the matrix $[\tilde{m}_{ij}(P)]$ where

$$\tilde{m}_{ij}(P) = \frac{1}{V(P)} \sum_{l=1}^{N} \frac{a_l}{h_l} (e_i \cdot u_l)(e_j \cdot u_l).$$

An alternate definition of the operator Γ_{-2} will be given after additional notation is introduced.

The easily established affine nature of the operator Γ_2 is formally stated in:

Lemma 1. *If $K \subset \mathbb{R}^n$ is star shaped about the origin, then for each $\phi \in \mathrm{GL}(n)$,*

$$\Gamma_2(\phi K) = \phi \Gamma_2 K.$$

While more difficult to see, we will prove

Lemma 1^***.** *Suppose* $K \subset \mathbb{R}^n$ *is a convex body that contains the origin in its interior. Then for each* $\phi \in \mathrm{GL}(n)$*,*

$$\Gamma_{-2}(\phi K) = \phi \Gamma_{-2} K.$$

The following theorem is fundamental and goes back, at least, to Blaschke [Bl], John [J], and Petty [P1] (see also Milman and Pajor [MPa1,MPa2]). We will give yet another proof in this paper.

Theorem 1. *If* $K \subset \mathbb{R}^n$ *is star shaped about the origin, then*

$$V(\Gamma_2 K) \geq V(K),$$

with equality if and only if K *is an ellipsoid centered at the origin.*

For our new ellipsoids we will establish:

Theorem 1^***.** *Suppose* $K \subset \mathbb{R}^n$ *is a convex body that contains the origin in its interior. Then*

$$V(\Gamma_{-2} K) \leq V(K),$$

with equality if and only if K *is an ellipsoid centered at the origin.*

The operator Γ_{-2} has the following monotonicity property:

Theorem 2^***.** *Suppose* $K \subset \mathbb{R}^n$ *is a convex body that contains the origin in its interior. If* E *is an ellipsoid centered at the origin such that* $E \subset K$*, then*

$$V(\Gamma_{-2} E) \leq V(\Gamma_{-2} K),$$

with equality if and only if $E = \Gamma_{-2} K$*.*

Let S^{n-1} denote the unit sphere, centered at the origin, in $\mathbb{R}^n$. Let B denote the unit ball, centered at the origin, in $\mathbb{R}^n$, and let $\omega_n = V(B)$.

From Theorem 2^* we will obtain:

Theorem 3^***.** *Suppose* $K \subset \mathbb{R}^n$ *is a convex body that is origin-symmetric, then*

$$V(\Gamma_{-2} K) \geq 2^{-n} \omega_n V(K),$$

with equality if and only if K *is a parallelotope.*

The analog of Theorem 3^* for the operator Γ_2 is one of the major open problems in the field: Finding the maximum of $V(\Gamma_2 K)/V(K)$ as K ranges even over the class of origin-symmetric convex bodies (or even important small subclasses) is difficult (see e.g., the survey of Lindenstrauss and Milman [LiM]). It is even difficult to show that there exists a c (independent of the dimension n) such that $[V(\Gamma_2 K)/V(K)]^{1/n}$ is bounded by $c\sqrt{n}$ as K ranges over the class of origin-symmetric convex bodies. This problem was first posed by Bourgain [Bo1]. The best known bounds to date appear to be those of Bourgain [Bo2] (see also Dar [D] and Junge [Ju]). There is an important class of questions in the *local theory of Banach spaces* which are well known to be equivalent in that an answer to one will immediately provide an answer to the others. Bourgain's problem is one member of this important class of equivalent problems. See Milman and Pajor [MPa2, Section 5]. We shall present a version of Theorem 3* for arbitrary convex bodies. We then present a classical characterization of the operator Γ_2 and its obvious counterpart for the operator Γ_{-2}. Finally, we shall present an analog of Milman's important notion of 'isotropic position' and explore some of its consequences. We have chosen to reprove all the classical results concerning the operator Γ_2 for two reasons. First we want to show the close connection and interrelationship between the operators Γ_2 and Γ_{-2}. Second, we believe that new proofs of classical results are almost always enlightening.

A serious attempt has been made to present all arguments in a reasonably self-contained manner. For quick reference, some basic properties of L_2-mixed and dual mixed volumes will be listed. Some recent applications of dual mixed volumes can be found in [G1], [Z1], [Z2] and [Z3]. The L_1-analogs of some of the identities presented may be found in [L1]. For general reference the reader may wish to consult the books of Gardner [G2], Schneider [S], and Thompson [T].

Recall that if $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior, then K^*, the polar of K, is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1, \quad \text{for all } y \in K\}.$$

From the definition it follows easily that for each convex body K, we have

$$K^{**} = K. \tag{1}$$

From the definition of a polar body, it follows trivially that for each convex body K and $\phi \in \mathrm{GL}(n)$

$$(\phi K)^* = \phi^{-t}(K^*), \tag{2}$$

where ϕ^{-t} denotes the inverse of the transpose of ϕ.

The radial function, $\rho_K = \rho(K, \cdot\,) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact, star–shaped (about the origin) $K \subset \mathbb{R}^n$, is defined, for $x \neq 0$, by

$$\rho(K, x) = \max\{\, \lambda \ge 0 : \lambda x \in K \,\}.$$

If ρ_K is positive and continuous, K is called a *star body* (about the origin). Two star bodies K and L are said to be *dilates* (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

From the definition of radial function, it follows immediately that for a star body K, an $x \in \mathbb{R}^n \setminus \{0\}$, and a $\phi \in \mathrm{GL}(n)$, we have

$$\rho_{\phi K}(x) = \rho_K(\phi^{-1}x), \tag{3}$$

$\phi K = \{\phi x \colon x \in K\}$ is the image of K under ϕ.

If $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior, then its support function, $h_K = h(K, \cdot\,) \colon \mathbb{R}^n \to (0, \infty)$, is defined for $x \in \mathbb{R}^n$ by

$$h(K, x) = \max\{x \cdot y \colon y \in K\}.$$

Since it is assumed throughout that all of our convex bodies contain the origin in their interiors, all support functions are strictly positive on $\mathbb{R}^n \setminus \{0\}$.

From the definition of support function, it follows immediately that for a convex body K, an $x \in \mathbb{R}^n$, and a $\phi \in \mathrm{GL}(n)$, we have

$$h_{\phi K}(x) = h_K(\phi^t x), \tag{3*}$$

where ϕ^t denotes the transpose of ϕ.

If K is a convex body, then it follows from the definitions of support and radial functions, and the definition of polar body, that

$$h_{K^*} = 1/\rho_K \qquad \text{and} \qquad \rho_{K^*} = 1/h_K. \tag{4}$$

For star bodies K, L, and $\varepsilon > 0$, the L_2-harmonic radial combination $K \,\widetilde{+}_{-2}\, \varepsilon \cdot L$ is the star body defined by

$$\rho(K \,\widetilde{+}_{-2}\, \varepsilon \cdot L, \cdot\,)^{-2} = \rho(K, \cdot\,)^{-2} + \varepsilon \rho(L, \cdot\,)^{-2}. \tag{5}$$

For convex bodies K, L, and $\varepsilon > 0$ the Firey L_2-combination $K +_2 \varepsilon \cdot L$ is defined as the convex body whose support function is given by

$$h(K +_2 \varepsilon \cdot L, \cdot\,)^2 = h(K, \cdot\,)^2 + \varepsilon h(L, \cdot\,)^2. \tag{5*}$$

Note that the "scalar" multiplication "$\varepsilon \cdot L$" in (5) and (5*) are different. The temptation to put a subscript under each "$\cdot$" was resisted.

From (4) we see that the relationship between the two types of combinations is that for convex bodies K, L, and $\varepsilon > 0$,

$$K +_2 \varepsilon \cdot L = (K^* \,\widetilde{+}_{-2}\, \varepsilon \cdot L^*)^*.$$

The dual mixed volume $V_{-2}(K,L)$ of the star bodies K,L, can be defined by

$$\frac{n}{-2}\,V_{-2}(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K\,\widetilde{+}_{-2}\,\varepsilon\cdot L)-V(K)}{\varepsilon}. \tag{6}$$

The L_2–mixed volume, $V_2(K,L)$, of the convex bodies K,L was defined in [L2] by:

$$\frac{n}{2}V_2(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K+_2\varepsilon\cdot L)-V(K)}{\varepsilon}. \tag{6*}$$

That this limit exists was demonstrated in [L2].

From the definitions (5) and (6), it follows immediately that for each star body K,

$$V_{-2}(K,K)=V(K). \tag{7}$$

From the definitions (5*) and (6*), it follows immediately that for each convex body K,

$$V_2(K,K)=V(K). \tag{7*}$$

From (3) and the definition of an L_2-harmonic radial combination (5) it follows immediately that for an L_2-harmonic radial combination of star bodies K and L,

$$\phi(K\,\widetilde{+}_{-2}\,\varepsilon\cdot L)=\phi K\,\widetilde{+}_{-2}\,\varepsilon\cdot\phi L.$$

This observation together with the definition of the dual mixed volume V_{-2} shows that for $\phi\in \mathrm{SL}(n)$ and star bodies K,L we have $V_{-2}(\phi K,\phi L)=V_{-2}(K,L)$ or equivalently

$$V_{-2}(\phi K,L)=V_{-2}(K,\phi^{-1}L). \tag{8}$$

From (3*) and the definition of a Firey L_2-combination (5*) it follows immediately that for a Firey combination of convex bodies K and L,

$$\phi(K+_2\varepsilon\cdot L)=\phi K+_2\varepsilon\cdot\phi L.$$

This observation together with the definition of the L_2-mixed volume V_2 shows that for $\phi\in \mathrm{SL}(n)$ and convex bodies K,L we have $V_2(\phi K,\phi L)=V_2(K,L)$ or equivalently

$$V_2(\phi K,L)=V_2(K,\phi^{-1}L). \tag{8*}$$

The definitions (5) and (6), and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $V_{-2}(K,L)$ of the star bodies K,L:

$$V_{-2}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n+2}(v)\rho_L^{-2}(v)\,dS(v), \tag{9}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}. It was shown in [L2], that corresponding to each convex body K, there is a positive Borel measure $S_2(K,\ \cdot\ )$ on S^{n-1} such that

$$V_2(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^2(u)\, dS_2(K,u), \tag{9*}$$

for each convex body L.

We will require two basic inequalities regarding the mixed volumes V_2 and the dual mixed volumes V_{-2}. The dual mixed volume inequality for V_{-2} is that for star bodies K, L,

$$V_{-2}(K,L) \geq V(K)^{(n+2)/n} V(L)^{-2/n}, \tag{10}$$

with equality if and only if K and L are dilates. This inequality is an immediate consequence of the Hölder inequality and the integral representation (9). The L_2-analog of the classical Minkowski inequality states that for convex bodies K, L,

$$V_2(K,L) \geq V(K)^{(n-2)/n} V(L)^{2/n}, \tag{10*}$$

with equality if and only if K and L are dilates. This L_2-analog of the Minkowski inequality was established in [L2] by using the classical Minkowski mixed volume inequality. An immediate consequence of the dual mixed volume inequality (10), and identity (7), that we shall use is the fact that if for star bodies K, L we have

$$V_{-2}(Q,K)/V(Q) = V_{-2}(Q,L)/V(Q),$$

for all star bodies Q, which belong to some class that contains both K and L, then in fact $K = L$.

It is easy to verify that if A is a positive definite $n \times n$ real symmetric matrix then the radial and support functions of the ellipsoid $\epsilon(A) = \{x \in \mathbb{R}^n : x{\cdot}Ax \leq 1\}$, are given by

$$\rho_{\epsilon(A)}^{-2}(u) = u \cdot Au \quad \text{and} \quad h_{\epsilon(A)}^2(u) = u \cdot A^{-1}u,$$

for $u \in S^{n-1}$. Thus, for a star body K,

$$h_{\Gamma_2 K}^2(u) = \frac{n+2}{V(K)}\int_K (u{\cdot}x)^2 dx, \tag{11}$$

for $u \in S^{n-1}$. The normalization above is chosen so that for the unit ball B, we have $\Gamma_2 B = B$. It must be emphasized that our normalization differs from the classical. For the polar of $\Gamma_2 K$ we will write $\Gamma_2^* K$ rather than $(\Gamma_2 K)^*$.

For each convex body K, we can define the ellipsoid $\Gamma_{-2}K$ by

$$\rho_{\Gamma_{-2}K}^{-2}(u) = \frac{1}{V(K)} \int_{S^{n-1}} (u \cdot v)^2 dS_2(K, v), \tag{11*}$$

for $u \in S^{n-1}$. Note that for the unit ball B, we have $\Gamma_{-2}B = B$. For the polar of $\Gamma_{-2}K$ we will write Γ_{-2}^*K rather than $(\Gamma_{-2}K)^*$, and thus

$$h_{\Gamma_{-2}^*K}^2(u) = \frac{1}{V(K)} \int_{S^{n-1}} (u \cdot v)^2 dS_2(K, v).$$

It was shown in [L2] that the L_2-surface area measure $S_2(K, \,\cdot\,)$ is absolutely continuous with respect to the classical surface area measure S_K and that the Radon-Nikodym derivative

$$\frac{dS_2(K, \,\cdot\,)}{dS_K} = 1/h_K.$$

Thus, if P is a polytope whose faces have outer unit normals $u_1, \dots, u_N$, and a_i denotes the area of the face with outer normal u_i and h_i denotes the distance from the origin to this face, then the measure $S_2(P, \,\cdot\,)$ is concentrated at the points $u_1, \dots, u_N \in S^{n-1}$ and $S_2(P, \{u_i\}) = a_i/h_i$. Thus, for the polytope P, we have

$$\rho_{\Gamma_{-2}P}^{-2}(u) = \frac{1}{V(P)} \sum_{l=1}^{N} (u \cdot u_l)^2 \frac{a_l}{h_l}$$

for $u \in S^{n-1}$.

If K is a convex body such that ∂K is C^2 and whose Gauss curvature is bounded, then it is well known that the measure S_K is absolutely continuous with respect to spherical Lebesgue measure (i.e., $(n-1)$-dimensional Hausdorff measure), S, and the Radon-Nikodym derivative

$$\frac{dS_K}{dS} = f_K$$

where $f_K \colon S^{n-1} \to (0, \infty)$ is the reciprocal Gauss curvature of ∂K viewed as a function of the outer normals (i.e., $f_K(u)$, for $u \in S^{n-1}$, is the reciprocal Gauss curvature at the point of ∂K whose outer unit normal is u). Thus for $u \in S^{n-1}$,

$$\rho_{\Gamma_{-2}K}^{-2}(u) = \frac{1}{V(K)} \int_{S^{n-1}} (u \cdot v)^2 h_K^{-1}(v) f_K(v)\, dS(v),$$

or equivalently,

$$\rho_{\Gamma_{-2}K}^{-2}(u) = \frac{1}{V(K)} \int_{\partial K} (u \cdot \nu(x))^2 h_K^{-1}(\nu(x))\, dx,$$

where $\nu(x)$ denotes the outer unit normal at $x \in \partial K$ and the integration is with respect to the intrinsic measure on ∂K.

A connection between the operators Γ_2 and Γ_{-2} is given in the following identity:

Lemma 2. *Suppose $K, L \subset \mathbb{R}^n$. If K is a convex body that contains the origin in its interior and L is a star body about the origin, then*

$$V_2(L, \Gamma_2 K)/V(L) = V_{-2}(K, \Gamma_{-2} L)/V(K).$$

Proof. From the integral representation (9*), definition (11), Fubini's theorem, definition (11*), and the integral representation (9), it follows that

$$\begin{aligned}
V_2(L, \Gamma_2 K) &= \frac{1}{n} \int_{S^{n-1}} h^2_{\Gamma_2 K}(u)\, dS_2(L, u) \\
&= \frac{1}{n} \int_{S^{n-1}} \left(\frac{n+2}{V(K)} \int_K (u \cdot x)^2 dx \right) dS_2(L, u) \\
&= \frac{1}{nV(K)} \int_{S^{n-1}} \int_{S^{n-1}} (u \cdot v)^2 \rho_K^{n+2}(v)\, dS(v)\, dS_2(L, u) \\
&= \frac{V(L)}{nV(K)} \int_{S^{n-1}} \rho_K^{n+2}(v) \rho^{-2}_{\Gamma_{-2} L}(v)\, dS(v) \\
&= \frac{V(L)}{V(K)} V_{-2}(K, \Gamma_{-2} L).
\end{aligned}$$

From the integral representation (9), definition (11), (4), and Fubini's theorem, we immediately see that if K and L are star bodies, then

$$V_{-2}(K, \Gamma_2^* L)/V(K) = V_{-2}(L, \Gamma_2^* K)/V(L) \tag{12}$$

From the integral representation (9*), definition (11*), (4), and Fubini's theorem, we immediately see that if K and L are convex bodies, then

$$V_2(K, \Gamma_{-2}^* L)/V(K) = V_2(L, \Gamma_{-2}^* K)/V(L) \tag{12*}$$

An immediate consequence of the definition of the L_2-centroid body (11) and the transformation rule for support function (3*), is that for $\phi \in \mathrm{GL}(n)$,

$$\Gamma_2 \phi K = \phi \Gamma_2 K.$$

Since, for the unit ball, B, we have $\Gamma_2 B = B$, it follows that if E is an ellipsoid centered at the origin, then

$$\Gamma_2 E = E.$$

The following lemma shows that Γ_{-2} is also an intertwining operator with the linear group $\mathrm{GL}(n)$.

Lemma 1^***.** *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior. If $\phi \in \mathrm{GL}(n)$, then*

$$\Gamma_{-2}(\phi K) = \phi \Gamma_{-2} K.$$

Proof. From Lemma 2, followed by (8^*), Lemma 1, Lemma 2 again, and (8) we have for each star body Q,

$$\begin{aligned} V_{-2}(Q, \Gamma_{-2}\phi K)/V(Q) &= V_2(\phi K, \Gamma_2 Q)/V(\phi K) \\ &= V_2(K, \phi^{-1}\Gamma_2 Q)/V(K) \\ &= V_2(K, \Gamma_2 \phi^{-1} Q)/V(K) \\ &= V_{-2}(\phi^{-1}Q, \Gamma_{-2}K)/V(\phi^{-1}Q) \\ &= V_{-2}(Q, \phi\Gamma_{-2}K)/V(Q). \end{aligned}$$

But $V_{-2}(Q, \Gamma_{-2}\phi K)/V(Q) = V_{-2}(Q, \phi\Gamma_{-2}K)/V(Q)$ for all star bodies Q implies that

$$\Gamma_{-2}\phi K = \phi \Gamma_{-2} K.$$

Since, for the unit ball, B, we have $\Gamma_{-2}B = B$, it follows from Lemma 1^* that if E is an ellipsoid centered at the origin, then

$$\Gamma_{-2}E = E.$$

Thus $\Gamma_{-2}\Gamma_2 K = \Gamma_2 K$, for all K. Now, in Lemma 2 take $L = \Gamma_2 K$, use (7^*), and get: For each star body K,

$$V_{-2}(K, \Gamma_2 K) = V(K). \tag{13}$$

But (13) and the dual mixed volume inequality (10) immediately yield:

Theorem 1. *If $K \subset \mathbb{R}^n$ is a star body about the origin, then*

$$V(\Gamma_2 K) \geq V(K),$$

with equality if and only if K is an ellipsoid centered at the origin.

In Lemma 2 take $K = \Gamma_{-2}L$, use (7), and get: For each convex body L,

$$V_2(L, \Gamma_{-2}L) = V(L). \tag{13*}$$

But (13^*) and the L_2-mixed volume inequality (10^*) immediately yield:

Theorem 1^***.** *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior. Then*

$$V(\Gamma_{-2}K) \leq V(K),$$

with equality if and only if K is an ellipsoid centered at the origin.

Theorem 2^***.** *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior. If E is an ellipsoid centered at the origin such that $E \subset K$, then*

$$V(\Gamma_{-2}E) \leq V(\Gamma_{-2}K),$$

with equality if and only if $E = \Gamma_{-2}K$.

Proof. From the integral representation (9^*) we see that the mixed volume $V_2(K, \cdot\,)$ is monotone with respect to set inclusion. Now From (7^*), the monotonicity of the mixed volume $V_2(K, \cdot\,)$, Lemma 1^* and (1), identity (12^*), and the L_2-mixed volume inequality (10^*), we have

$$\begin{aligned} 1 &= V_2(K,K)/V(K) \\ &\geq V_2(K,E)/V(K) \\ &= V_2(K,\Gamma^*_{-2}E^*)/V(K) \\ &= V_2(E^*,\Gamma^*_{-2}K)/V(E^*) \\ &\geq [V(E^*)/\omega_n]^{-2/n}[V(\Gamma^*_{-2}K)/\omega_n]^{2/n} \\ &= [\omega_n/V(E)]^{-2/n}[\omega_n/V(\Gamma_{-2}K)]^{2/n}, \end{aligned}$$

where the last equality is a consequence of the fact that, by (2), the product of the volumes of polar reciprocal ellipsoids, that are centered at the origin, is ω_n^2. Hence we have,

$$V(\Gamma_{-2}K) \geq V(E) = V(\Gamma_{-2}E),$$

with equality (from the equality conditions of the L_2-mixed volume inequality (10^*)) implying that E and $\Gamma_{-2}K$ are dilates, which in turn implies that $E = \Gamma_{-2}K$. □

The infimum of $V(\Gamma_{-2}K)/V(K)$ taken over all convex bodies that contain the origin in their interiors is 0. To get a positive lower bound some restriction must be made on the positions of the bodies (relative to the origin).

A fundamental result due to Ball [B] is that if K is an origin-symmetric convex body, then there exists an ellipsoid $E_K \subset K$, centered at the origin, such that $V(E_K) \geq 2^{-n}\omega_n V(K)$. Barthe [Br] improved Ball's theorem by showing that for each origin-symmetric convex body K that is not a parallelotope, there exists an ellipsoid $E_K \subset K$, centered at the origin, such that

$$V(E_K) > 2^{-n}\omega_n V(K).$$

Combine this with Theorem 2^* and the immediate result is:

Theorem 3^***.** *Suppose $K \subset \mathbb{R}^n$ is a convex body that is origin-symmetric, then*

$$V(\Gamma_{-2}K) \geq \frac{\omega_n}{2^n} V(K),$$

with equality if and only if K is a parallelotope.

Associated with each convex body K is an important affinely associated point called the John point, $j(K) \in \operatorname{int} K$. This point is the center of the (unique) ellipsoid of maximal volume that is contained in the body K. (As an aside, the authors note that in their opinions it would be more appropriate to call this point the Löwner point.) The John point is an affinely associated point in that for each $\phi \in \mathrm{SL}(n)$ we have $j(\phi K) = \phi j(K)$.

A fundamental result due to Ball [B] is that if K is positioned so that its John point is at the origin, then there exists an ellipsoid $E_K \subset K$, centered at the origin, such that $V(E_K) \geq n!\,\omega_n\, n^{-n/2}(n+1)^{-(n+1)/2} V(K)$. Barthe [Br] improved Ball's theorem by showing that if K is positioned so that its John point is at the origin, then unless K is a simplex, there exists an ellipsoid $E_K \subset K$, centered at the origin, such that

$$V(E_K) > \frac{n!\,\omega_n}{n^{n/2}(n+1)^{(n+1)/2}} V(K).$$

Together with Theorem 2* this immediately gives:

Theorem 4*. *If $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, then*

$$V(\Gamma_{-2}K) \geq \frac{n!\,\omega_n}{n^{n/2}(n+1)^{(n+1)/2}} V(K),$$

with equality if and only if K is a simplex.

The volume-normalized version of the operator Γ_2 is the operator that maps each star body K to $[\omega_n/V(\Gamma_2 K)]^{1/n}\Gamma_2 K$. A classical characterization of the volume-normalized version of the operator Γ_2 is as the solution to the following problem: Given a fixed star body K, find an ellipsoid centered at the origin, E, that minimizes $V_{-2}(K, E)$ subject to the constraint that $V(E) = \omega_n$. Existence, uniqueness, and characterization of the solution to the problem are all contained in:

Theorem 5. *Suppose $K \subset \mathbb{R}^n$ is a star body about the origin and E is an ellipsoid centered at the origin such that $V(E) = \omega_n$. Then*

$$V_{-2}(K, E) \;\geq\; V(K)\,[V(\Gamma_2 K)/\omega_n]^{2/n},$$

with equality if and only if $E = \lambda \Gamma_2 K$ where $\lambda = [\omega_n/V(\Gamma_2 K)]^{1/n}$.

Proof. From (2) and Lemma 1, followed by (12), and the dual Minkowski inequality (10), we have

$$\begin{aligned} V_{-2}(K, E)/V(K) &= V_{-2}(K, \Gamma_2^* E^*)/V(K) \\ &= V_{-2}(E^*, \Gamma_2^* K)/V(E^*) \\ &\geq \omega_n^{-2/n} V(\Gamma_2 K)^{2/n}, \end{aligned}$$

with equality if and only if E and $\Gamma_2 K$ are dilates.

Suppose $K \subset \mathbb{R}^n$ is a fixed convex body that contains the origin in its interior. Find an ellipsoid, E, centered at the origin, which minimizes $V_2(K, E)$ subject to the constraint that $V(E) = \omega_n$. The solution of the problem turns out to characterize the volume-normalized operator Γ_{-2}. Existence, uniqueness, and characterization of the solution to the problem are all contained in:

Theorem 5^***.** *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior and E is an ellipsoid centered at the origin such that $V(E) = \omega_n$. Then*

$$V_2(K,E) \;\geq\; V(K)\,[V(\Gamma_{-2}K)/\omega_n]^{-2/n},$$

with equality if and only if $E = \lambda\Gamma_{-2}K$ where $\lambda = [\omega_n/V(\Gamma_{-2}K)]^{1/n}$.

Proof. From (2) and Lemma 1^*, followed by (12^*), and the L_2-Minkowski inequality (10^*), we have

$$\begin{aligned} V_2(K,E)/V(K) &= V_2(K,\Gamma^*_{-2}E^*)/V(K) \\ &= V_2(E^*,\Gamma^*_{-2}K)/V(E^*) \\ &\geq \omega_n^{-2/n} V(\Gamma^*_{-2}K)^{2/n}, \end{aligned}$$

with equality if and only if E and $\Gamma_{-2}K$ are dilates.

The L_1-analog of the problem solved by Theorem 5^* was treated by Petty [P2]. Generalizations were considered by Clack [C] and Giannopoulos and Papadimitrakis [GiPap].

A star body K is said to be in *isotropic* position if $\Gamma_2 K$ is a ball and $V(K) = 1$. Note that for each star body there is a $GL(n)$-transformation that will map the body into one that is in isotropic position. If the star body, K, is in isotropic position, then the isotropic constant, L_K, of K is defined to be the radius of $\frac{1}{\sqrt{n+2}}\Gamma_2 K$. If K is an arbitrary star body, then define its isotropic constant by

$$L_K = \frac{1}{\sqrt{n+2}}\left[\frac{V(\Gamma_2 K)}{\omega_n V(K)}\right]^{1/n}.$$

From Theorem 1, it immediately follows that for each star body K,

$$L_K \geq \frac{\omega_n^{-1/n}}{\sqrt{n+2}},$$

with equality if and only if K is an ellipsoid centered at the origin. An important question (previously mentioned) asks if

$$\sup\{L_K \colon K \text{ is a convex body in } \mathbb{R}^n \text{ in isotropic position}\}$$

is dominated by a real number independent of the dimension n.

A convex body K will be said to be in *dual isotropic* position if $\Gamma_{-2}K$ is a ball and $V(K) = 1$. Note that for each convex body there is a $GL(n)$-transformation that will map the body into one that is in isotropic position.

If K is in dual isotropic position, then define the dual isotropic constant, L_K^*, to be the radius of $\Gamma_{-2}K$. If K is an arbitrary convex body we can define its dual isotropic constant by

$$L_K^* = \left[\frac{V(\Gamma_{-2}K)}{\omega_n V(K)}\right]^{1/n}.$$

Theorems 1* and 3* immediately give:

Theorem 6*. *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior. If K origin-symmetric and in dual isotropic position, then*

$$\tfrac{1}{2} \le L_K^* \le \omega_n^{-1/n}.$$

Equality on the left-hand side holds if and only if K is a parallelotope and equality on the right-hand side holds if and only if K is an ellipsoid.

Let v denote $(n-1)$-dimensional volume. For $u \in S^{n-1}$, let $u^\perp$ denote the 1-codimension subspace of $\mathbb{R}^n$ that is orthogonal to u. Milman and Pajor [MPa2] showed that if K is origin-symmetric, then

$$\tfrac{\sqrt{n+2}}{2\sqrt{3}} \frac{V(K)}{\mathrm{v}(K \cap u^\perp)} \le h_{\Gamma_2 K}(u) \le \tfrac{n}{\sqrt{2(n+2)}} \frac{V(K)}{\mathrm{v}(K \cap u^\perp)}$$

for all $u \in S^{n-1}$. Equality on the left-hand side holds for K a right cylinder and u orthogonal to its base, and equality on the right-hand side holds for K a double right cone and u along its axis.

For $u \in S^{n-1}$ and a convex body K, let $K|u^\perp$ denote the image of the orthogonal projection of K onto $u^\perp$.

Theorem 7*. *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior. If K is origin-symmetric, then for every $u \in S^{n-1}$*

$$\tfrac{2}{\sqrt{n}} \frac{\mathrm{v}(K|u^\perp)}{V(K)} \le h_{\Gamma^*_{-2}K}(u) \le 2\frac{\mathrm{v}(K|u^\perp)}{V(K)}.$$

Equality on the left-hand side holds for K a double right cone and u along its axis, and equality on the right-hand side holds for K a right cylinder and u orthogonal to its base.

Proof. From (11*), together with the fact that $dS_2(K,\ \cdot\ ) = h_K^{-1} dS_K$, and the Hölder inequality we have

$$\begin{aligned}\rho_{\Gamma_{-2}K}^{-1}(u) &= \left[\frac{1}{nV(K)}\int_{S^{n-1}}\left(\frac{\sqrt{n}|u\cdot v|}{h_K(v)}\right)^2 h_K(v)dS_K(v)\right]^{1/2}\\ &\geq \frac{1}{nV(K)}\int_{S^{n-1}}\sqrt{n}|u\cdot v|dS_K(v)\\ &= \frac{2\mathrm{v}(K|u^\perp)}{\sqrt{n}V(K)},\end{aligned}$$

which gives the left inequality.

To get the right-hand inequality, note that

$$\begin{aligned}\rho_{\Gamma_{-2}K}^{-1}(u) &= \left[\frac{1}{V(K)}\int_{S^{n-1}}\frac{|u\cdot v|}{h_K(v)}|u\cdot v|dS_K(v)\right]^{1/2}\\ &\leq \left[\frac{2\mathrm{v}(K|u^\perp)}{V(K)}\max_{v\in S^{n-1}}\frac{|u\cdot v|}{h_K(v)}\right]^{1/2}\\ &= \left[\frac{2\mathrm{v}(K|u^\perp)}{V(K)}\rho_K(u)^{-1}\right]^{1/2}\\ &\leq \frac{2\mathrm{v}(K|u^\perp)}{V(K)},\end{aligned}$$

where the last inequality follows from the well-known and easily-established fact that $V(K) \leq 2\mathrm{v}(K|u^\perp)\rho_K(u)$.

Theorem 8*. *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior. If K is origin-symmetric and in dual isotropic position, then*

$$\mathrm{v}(K|u^\perp) \leq \sqrt{n},$$

for all $u \in S^{n-1}$. Equality holds if and only if K is a cube and u is in the direction of one of its vertices.

Proof. Suppose K is not a cube. From the left-hand inequality of Theorem 7*, the fact that $[V(\Gamma_{-2}K)/\omega_n]^{1/n}$ is the radius of the ball $\Gamma_{-2}K$, together with Theorem 3*, we have

$$\frac{2\mathrm{v}(K|u^\perp)}{\sqrt{n}V(K)} \leq \left[\frac{V(\Gamma_{-2}K)}{\omega_n}\right]^{-1/n} < 2V(K)^{-1/n}. \quad \square$$

Keith Ball conjectured that each origin-symmetric convex body can be $GL(n)$-transformed into a body for which the inequality of Theorem 8^* holds. Giannopoulos and Papadimitrakis [GiPap] showed that this can be accomplished by making the body "surface isotropic". Theorem 8^* shows that this can also be done by making the body dual isotropic.

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