CURVATURE AND THE THEOREMA EGREGIUM OF GAUSS

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In this note, we describe a simple way to define the second fundamental form of a hypersurface in $\mathbb{R}^n$ and use it to prove Gauss's Theorema Egregium, as well as its analogue in higher dimensions. The proof can be extended to submanifolds of higher codimension.

1. Rigid motions

Let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$. Given a nonzero vector $\nu \in \mathbb{R}^n$, let $\nu^\perp$ denote the linear hyperplane normal to ν.

A *rigid motion* is a map $\Phi \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(x) = Ax + b,$$

where $A \in \mathrm{SO}(n)$ and $b \in \mathbb{R}^n$.

A simple fact that we will use is the following: Given any $x \in \mathbb{R}^n$ and hyperplane P passing through x, there exists a rigid motion R such that $R(x) = 0$ and $R(P) = e_n^\perp$. The rigid motion is unique up to a rotation of $e_n^\perp$.

2. Definition of a hypersurface

A *hypersurface* is a subset $S \subset \mathbb{R}^n$ such that for each $x \in S$, there exists a hyperplane T passing through x such that a neighborhood of x in S is graph of a smooth function over the plane T.

In other words, S is a hypersurface if for any $x \in S$, there exists a hyperplane T_x containing x with normal ν and a continuous function $f : T_x \to \mathbb{R}$ such that

$$f(x) = 0,$$

and there exists a neighborhood $D \subset T_x$ of x such that

$$y + f(y)\nu \in S \text{ for each } y \in D.$$

If $df(x) = 0$, then we call ν a *normal* and T the *tangent plane* to S at x. If $|\nu| = 1$, then it is called a *unit normal*.

If S is a hypersurface and $x \in S$, then the tangent plane T_x is unique. The unit normal ν and therefore the function f restricted to a sufficiently small neighborhood of $x \in T$ are unique up to sign.

Given any $x \in S$ and unit normal ν at x, there is a rigid motion R that moves x to the origin and ν to e_3. In particular, $R(S)$ is locally the graph of a function f over the horizontal plane $e_n^\perp$. The rigid motion and therefore the function f are uniquely defined up to a rotation of $e_n^\perp$. It follows that any function of f and its derivatives, evaluated at x, that is invariant under rotations of $e_n^\perp$ and changes of the sign of f is a pointwise geometric invariant of S.

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3. The second fundamental form

We can therefore assume below that $x = 0$ and $\nu = e_n$, and $T = e_n^\perp$. Since S is the graph of f in a neighborhood of 0, it is locally the image of the smooth embedding

$$y \mapsto y + f(x)e_n, y \in T.$$

Since the tangent plane T is horizontal at $x = 0$,

$$df(0) = 0.$$

Since $f(0) = 0$ and $df(0) = 0$, the simplest geometric invariants of S are defined in terms of the Hessian of f at 0, which is uniquely defined up to rotations of T. It follws that any function of the eigenvalues of $\partial^2(0)f$ defines a local geometric invariant, up to sign, of S. Any even function is a geometric invariant. In particular, if $n = 2$, the determinant of $\partial^2 f(0)$ is a local geometric invariant known as Gauss curvature.

The pullback of the Hessian to T_x is the second fundamental form of S at x.

4. Isometric hypersurfaces

Two hypersurfaces S and $\widehat{S}$ are *isometric* if there exists a smooth diffeomorphism

$$\Phi : S \to \widehat{S}$$

such that the length of any smooth curve $C \subset S$ is equal to the length of the curve $\Phi(C) \subset \widehat{S}$.

If $\hat{x} = \Phi(x)$, we can always move the two hypersurfaces, as described above, so that $\hat{x} = x = 0$, $T_{\hat{x}} = T_x = T$, S is the graph of a function f and $\widehat{S}$ the graph of another function $\hat{f}$. Moreover, there is a diffeomorphism $\phi : T \to T$ such that

$$\Phi(y + f(y)e_n) = \phi(y) + \hat{f}(\phi(y))e_n.$$

The diffeomoprhism $\Phi : S \to \widehat{S}$ preserves the lengths of all curves if and only if

$$\partial_i\hat{y} \cdot \partial_j\hat{y} = \partial_i y \cdot \partial_j y,$$

where

$$\begin{aligned} \hat{y} &= \phi^k(x)e_k + \hat{f}(\phi(x))e_n \\ y &= x^i e_i + f(x)e_n, \end{aligned}$$

for each $x \in T$. In other words,

$$\partial_i\phi \cdot \partial_j\phi + \partial_p\hat{f}\partial_i\phi^p\partial_q\hat{f}\partial_j\phi^q = \delta_{ij} + \partial_i f\partial_j f. \tag{1}$$

Note that

$$\begin{aligned} \partial_i f(0) &= \partial_i\hat{f}(0) = 0 \\ \partial_i\phi^j(0) &= \delta_i^j, \end{aligned}$$

for all $1 \leq i, j \leq n - 1$. Therefore, if we differentiate (1) and evaluate at $x = 0$, we get

$$\partial^2_{ik}\phi^j + \partial^2_{jk}\phi^i = 0.$$

It follows, by the usual argument,

$$\partial^2_{ik}\phi^j = -\partial^2_{jk}\phi^i = \partial^2_{ji}\phi^k = -\partial^2_{ik}\phi^j, \tag{2}$$

that $\partial^2_{jk}\phi^i(0) = 0$. If we differentiate (1) again, evaluate at $x = 0$, and cycle through the indices i, j, k, l, we get

$$\partial^3_{ikl}\phi^j + \partial^3_{jkl}\phi^i + \partial^2_{ik}\hat{f}\partial^2_{jl}\hat{f} + \partial^2_{il}\hat{f}\partial^2_{jk}\hat{f} = \partial^2_{ik}f\partial^2_{jl}f + \partial^2_{il}f\partial^2_{jk}f \tag{3}$$

$$\partial^3_{jli}\phi^k + \partial^3_{kli}\phi^j + \partial^2_{jl}\hat{f}\partial^2_{ki}\hat{f} + \partial^2_{ji}\hat{f}\partial^2_{kl}\hat{f} = \partial^2_{jl}f\partial^2_{ki}f + \partial^2_{ji}f\partial^2_{kl}f \tag{4}$$

$$\partial^3_{kij}\phi^l + \partial^3_{lij}\phi^k + \partial^2_{ki}\hat{f}\partial^2_{lj}\hat{f} + \partial^2_{kj}\hat{f}\partial^2_{li}\hat{f} = \partial^2_{ki}f\partial^2_{lj}f + \partial^2_{kj}f\partial^2_{li}f \tag{5}$$

$$\partial^3_{ljk}\phi^i + \partial^3_{ijk}\phi^l + \partial^2_{lj}\hat{f}\partial^2_{ik}\hat{f} + \partial^2_{lk}\hat{f}\partial^2_{ij}\hat{f} = \partial^2_{lj}f\partial^2_{ik}f + \partial^2_{lk}f\partial^2_{ij}f. \tag{6}$$

Therefore,

$$(3) - (4) + (5) - (6) \tag{7}$$

eliminates ϕ and gives

$$\partial^2_{ik}\hat{f}\partial^2_{jl}\hat{f} - \partial^2_{il}\hat{f}\partial^2_{ik}\hat{f} = \partial^2_{ik}f\partial^2_{jl}f - \partial^2_{il}f\partial^2_{ik}f. \tag{8}$$

It follows that if $e_1, \dots, e_n$ of T_x comprise an orthonormal basis of T_x, $\hat{e}_i = \Phi_* e_i$, $1 \le i \le n$, the corresponding orthonormal basis of $T_{\hat{x}}$, and

$$H_{ij} = \partial^2_{ij}f(0)$$
$$\widehat{H}_{ij} = \partial^2_{ij}\hat{f}(0),$$

are the second fundamental forms of S and $\widehat{S}$, then

$$\widehat{R}_{ijkl} = R_{ijkl},$$

where

$$R_{ijkl} = H_{ik}H_{jl} - H_{il}H_{jk}$$
$$\widehat{R}_{ijkl} = \widehat{H}_{ik}\widehat{H}_{jl} - \widehat{H}_{il}\widehat{H}_{jk}$$

These are, of course, the Riemann curvature tensors of S and $\widehat{S}$. That they are the same for the two surfaces, which have the same intrinsic but not necessarily the same extrinsic geometry, shows that the Riemann curvature tensor is an intrinsic geometric invariant for hypersurfaces in $\mathbb{R}^n$.

If $n = 2$, then this proves the Theorema Egregium of Gauss, because the only nonzero component of R is the Gauss curvature $K = R_{1212} = \det H$.

5. Tensor identities

Most of the proof above involves only differentiation and straightforward calculations. The only significant steps are the following:

(1) Definition of a hypersurface
(2) If a map between two hypersurfaces preserves lengths of curves, then the map satisfies (1).
(3) Most importantly, the tensor calculations done in (2) and (7). These are equivalent to the following tensor identities:

$$(T \otimes S^2T) \cap (\Lambda^2 T \otimes T) = \{0\}$$
$$(T \otimes S^3T) \cap (\Lambda^2 T \otimes \Lambda^2 T) = \{0\}.$$

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