HOLONOMY EQUALS CURVATURE

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Given a point p in a Riemannian manifold M, a closed curve γ that starts and ends at the point p, and a tangent vector $v \in T_pM$, let $P_\gamma v \in T_pM$ denote the vector obtained by parallel transporting v along γ. This defines a rotation $P_\gamma : T_pM \to T_pM$. The set of all rotations P_γ obtained via closed curves starting and ending at p is called the *holonomy group at p*.

We show here that the curvature tensor at p can be reconstructed from holonomy along small loops starting from p. In particular, if a loop ℓ lies in a sufficiently small neighborhood of p, then

$$\frac{R(X,Y)Z}{|X \wedge Y|} \simeq \frac{P_\gamma Z - Z}{\text{area}(D)},$$

where D is the disk comprised of geodesic segments joining p to each point in the loop.

This follows from the following.

Lemma 1. *Let $D \subset M \backslash C_p$, where C_p is the cut locus of p, be a smooth closed 2-disk such that $p \in \partial D$ and D is foliated by connected geodesic segments starting from p. Then*

$$P_\gamma Z_p - Z_p = \int_D \frac{\tau(R(X,Y)Z)}{|X \wedge Y|}\, dA, \tag{1}$$

where

- *dA is the surface area measure on D induced by the Riemannian metric on M.*
- *X and Y are linearly independent tangent vector fields on D.*
- *$\gamma : [0,1] \to \partial D$ is a parameterization of ∂D such that $\gamma(0) = \gamma(1) = p$, and, given any inward pointing vector $S \in T_pD$, the orientation of $(\gamma'(0), S)$ is the same as (X,Y).*
- *$Z_p \in T_pM$ and Z is defined by parallel translating Z_p first along the parameterized curve γ and then, for each $0 \le t \le 1$, along the unique geodesic segment going from $\gamma(t) \in \partial D$ to p.*
- *τ is parallel translation from each point in D to p along the unique geodesic segment joining them.*

Proof. Define a map $\Gamma : [0,1] \times [0,1] \to D$ such that, for each $0 \le t \le 1$, $\Gamma(1,t) = \gamma(t)$ and $\Gamma(\cdot,t)$ is a constant speed parameterization of the geodesic segment joining p to $\gamma(t)$. In particular, $\Gamma([0,1] \times [0,1]) = D$.

All vector fields below should be viewed as sections of the pullback bundle $\Gamma^* T_* M$ and therefore functions of $(s,t) \in [0,1] \times [0,1]$. This is a simple but useful trick.

Denote $S = \partial_s \Gamma$ and $T = \partial_t \Gamma$. Note that $[S,T] = 0$ and T is a Jacobi field along each geodesic $\Gamma(\cdot,t)$, $0 \le t \le 1$. Let

$$J = T - \frac{S \cdot T}{|S|^2} S,$$

which is a nonvanishing Jacobi field perpendicular Observe that

$$dA = |S \wedge T|\, ds\, dt = |S||J|\, ds\, dt.$$

Choose an orthonormal frame $e_1, \dots, e_n \in T_pM$ and extend by parallel translation along each geodesic $\Gamma(\cdot, t)$, $t \in [0,1]$. In particular, for each $(s,t) \in [0,1] \times [0,1]$ and $1 \le i \le n$,

$$\begin{aligned} \nabla_T e_i(0,t) &= 0 \\ \nabla_S e_i(s,t) &= 0. \end{aligned}$$

By definition, the vector field Z satisfies for each $(s,t) \in [0,1] \times [0,1]$,

$$\begin{aligned} \nabla_T Z(1,t) &= 0 \\ \nabla_S Z(s,t) &= 0. \end{aligned}$$

Observe that $Z_p = Z(0,0) = Z(1,0)$ and and $P_\gamma Z_p = Z(1,1) = Z(0,1)$, since $\Gamma(\cdot, 0)$ and $\Gamma(\cdot, 1)$ are just constant curves that stay at p. Therefore,

$$\begin{aligned} e_i(p) \cdot (P_\gamma Z_p - Z_p) &= e_i(0,1) \cdot Z(0,1) - e_i(0,0) \cdot Z(0,0) \\ &= \int_0^1 \partial_t (e_i(0,t) \cdot Z(0,t))\, dt \\ &= \int_0^1 e_i \cdot \nabla_T Z(0,t)\, dt \\ &= \int_0^1 \left[e_i \cdot \nabla_T Z(1,t) - \int_0^1 \partial_s (e_i \cdot \nabla_T Z(s,t))\, dx \right] dt \\ &= -\int_0^1 \int_0^1 e_i \cdot \nabla_S \nabla_T Z(s,t)\, ds\, dt \\ &= -\int_0^1 \int_0^1 e_i \cdot R(S,T) Z(s,t)\, ds\, dt \\ &= -\int_0^1 \int_0^1 e_i \cdot R(S,J) Z(s,t)\, ds\, dt \\ &= -\int_0^1 \int_0^1 e_i \cdot R(\sigma,\tau) Z |S||J|\, ds\, dt \\ &= \int_\Omega \frac{e_i \cdot R(X,Y)Z}{|X \wedge Y|}\, dA_g, \end{aligned}$$

where $\sigma = S/|S|$, $\tau = J/|J|$ form an oriented orthonormal frame along D. Since, for each $(s,t) \in [0,1] \times [0,1]$,

$$\tau(R(X,Y)Z) = \sum_{i=1}^n e_i(0,0)(e_i(s,t) \cdot R(X,Y)Z),$$

the lemma follows. ∎

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