

# HOLONOMY IS CURVATURE

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Let  $E$  be a vector bundle over a smooth manifold  $M$  and  $\nabla$  a connection on  $E$ . The curvature of the connection is the section  $\Omega$  of  $\bigwedge^2 T^*M \otimes \text{Aut}(E)$  such that

$$(1) \quad \Omega(X, Y)e = ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})e \in E_x,$$

for any  $x \in M$ ,  $X, Y \in T_x M$ ,  $e \in E_x$ .

Given a smooth curve  $c : [0, 1] \rightarrow M$ , the parallel transport of  $e \in E_{c(0)}$  along  $c$  is defined to be the section  $f : [0, 1] \rightarrow E$  such that the following hold for each  $t \in [0, 1]$ :

$$\begin{aligned} f(t) &\in E_{c(t)} \\ f(0) &= e \\ \nabla_T f(t) &= 0, \end{aligned}$$

where  $T = \partial_t$ . Denote  $P_c e = f(1)$ .

Let  $c : [0, 1] \rightarrow M$  be a  $C^1$  null-homotopic curve based at  $x$ . There exists a  $C^1$  map  $C : [0, 1] \times [0, 1] \rightarrow M$  satisfying the following for each  $0 \leq s, t \leq 1$ :

$$\begin{aligned} C(0, t) &= x \\ C(1, t) &= c(t) \\ C(s, 0) &= x \\ C(s, 1) &= x \end{aligned}$$

Given  $e_x \in E_x$ , let  $e : [0, 1] \times [0, 1] \rightarrow E$  be  $C^2$  section of  $C^*E$  satisfying the following for all  $0 \leq s, t \leq 1$ :

$$\begin{aligned} e(s, t) &\in E_{C(s, t)} \\ e(s, 0) &= e_x \\ \nabla_T e(1, t) &= 0 \\ \nabla_S e(s, t) &= 0, \end{aligned}$$

where  $S = \partial_s$  and  $T = \partial_t$ . In particular,

$$e(s, 1) = P_c e_x.$$

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Let  $E^*$  be the dual vector bundle of  $E$ . Given  $\varepsilon_x \in E_x^*$ , let  $\varepsilon : [0, 1] \times [0, 1] \rightarrow E^*$  satisfy the following for all  $0 \leq s, t \leq 1$ :

$$\begin{aligned}\varepsilon(s, t) &\in E_{C(s,t)}^* \\ \varepsilon(0, t) &= \varepsilon_x \\ \varepsilon(s, 0) &= \varepsilon_x \\ \varepsilon(s, 1) &= \varepsilon_x \\ \nabla_S \varepsilon(s, t) &= 0.\end{aligned}$$

It follows that

$$\nabla_T \varepsilon(0, t) = 0.$$

**Lemma 1.**

$$\langle \varepsilon_x, P_c e_x - e_x \rangle = \int_{[0,1] \times [0,1]} \langle \varepsilon(s, t), C^* \Omega \rangle.$$

*Proof.*

$$\begin{aligned}\langle \varepsilon_x, P_c e_x - e_x \rangle &= \langle \varepsilon(0, 1), e(0, 1) \rangle - \langle \varepsilon(0, 0), e(0, 0) \rangle \\ &= \int_{t=0}^{t=1} \partial_t (\langle \varepsilon(0, t), e(0, t) \rangle) dt \\ &= \int_{t=0}^{t=1} \langle \varepsilon, \nabla_T e(0, t) \rangle dt \\ &= \int_{t=0}^{t=1} \left[ \langle \varepsilon, \nabla_T e(1, t) \rangle - \int_{s=0}^{s=1} \partial_s (\langle \varepsilon, \nabla_T e(s, t) \rangle) \right] ds dt \\ &= - \int_{t=0}^{t=1} \int_{s=0}^{s=1} \langle \varepsilon, \nabla_S \nabla_T e(s, t) \rangle ds dt \\ &= \int_{t=0}^{t=1} \int_{s=0}^{s=1} \langle \varepsilon, \Omega(C_* T, C_* S) e(s, t) \rangle ds dt \\ &= \int_{[0,1] \times [0,1]} \langle \varepsilon, C^* \Omega \rangle e.\end{aligned}$$

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