HOLONOMY EQUALS CURVATURE

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Let E be a vector bundle over a manifold M, $\mathrm{Aut}(E)$ the space of automorphisms of E, and ∇ a connection on E. The curvature 2-form of ∇ at p is defined to be the $\mathrm{Aut}(E)$-valued 2-form Ω, where, if $p \in M$, $X, Y \in T_pM$, and s is a section of E over a neighborhood of p,

$$\langle \Omega, X \otimes Y \rangle s = R(X, Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s.$$

1. Pullback connection

Let N be a smooth manifold and $f : N \to M$ a smooth map. The pullback of the vector bundle E is defined to be the unique bundle f^*E, where there is a natural map

$$\begin{aligned} f_* : \Gamma(E) &\to \Gamma(f^*E) \\ s &\mapsto f_* s = s \circ f. \end{aligned}$$

Given a connection ∇ on E, the pullback of ∇ is the unique connection on f^*E such that, for any $s \in \Gamma(E)$,

$$\nabla_V f_* s = \nabla_{f_* V} s$$

2. Holonomy groups

Given a point p in a manifold M, a closed curve γ that starts and ends at the point p, and $v \in E_p$, where E_p denotes the fiber of E at p. let $P_\gamma v \in E_p$ denote the vector obtained by parallel transporting v along γ. This defines a linear map $P_\gamma : E_p \to E_p$.

Given an open set $O \subset M$, we can define the holonomy group of O with respect to the connection ∇ to be

$$H_O = \{P_\gamma \ : \ \gamma \subset O\}$$

and the *local holonomy group* at $p \in M$ to be

$$H_p = \bigcap_{p \in O} H_O.$$

Theorem 1. *For each $p \in M$,*

$$H_p = \{R(X, Y) : E_p \to E_p \ : \ X, Y \in T_pM\}.$$

3. Loop formula

Theorem 1 is a consequence of the following:

Theorem 2. *If*

$$\begin{aligned} M &= \textit{smooth manifold} \\ E &= \textit{rank } n \textit{ vector bundle over } M \\ \nabla &= \textit{connection on } E \\ R &= \textit{curvature tensor of } \nabla \\ \Omega &= \textit{curvature 2-form of } \nabla \\ \gamma : [0,1] \to M &= \textit{ null-homotopic loop in } M \\ p &= \gamma(0) = \gamma(1) \\ \Gamma : [0,1] \times [0,1] \to M &= \textit{ smooth homotopy from } p \textit{ to } \gamma \\ P_\gamma : E_p \to E_p &= \textit{ parallel translation around } \gamma, \end{aligned}$$

then

$$\int_{[0,1]\times[0,1]} \pi^{-1}_{(s,t)} \Omega \hat{\pi}_{(s,t)} = P_\gamma - 1,$$

where $\pi_{(s,t)} : E_p \to E_{\Gamma(s,t)}$ *is parallel translation along the curve* $\Gamma(\cdot, t)$ *from* p *to* $\Gamma(s,t)$ *and* $\hat{\pi}_{(s,t)} : E_p \to E_{\Gamma(s,t)}$ *is parallel translation first along* $\Gamma(1,\cdot)$ *from* p *to* $\gamma(t) = \Gamma(1,t)$ *and then along* $\Gamma(\cdot,t)$ *from* $\Gamma(1,t)$ *to* $\Gamma(s,t)$.

Proof. Assume that $\Gamma : [0,1] \times [0,1] \to M$ is a smooth map satisfying, for each $0 \le s \le 1$,

$$\begin{aligned} \Gamma(s,0) &= p \\ \Gamma(s,1) &= \gamma(s). \end{aligned}$$

All sections below should be viewed as sections of the pullback bundle Γ^*E and therefore functions of $(s,t) \in [0,1] \times [0,1]$. The connection and curvature are also assumed to be pulled back by Γ to $[0,1] \times [0,1]$ This is a simple but useful trick.

Denote $S = \partial_s$ and $T = \partial_t$. Note that $[S,T] = 0$.

Given $\omega_p \in E^*_p$, extend it by parallel translation along each curve $\Gamma(s,\cdot)$, $s \in [0,1]$. In particular, for each $(s,t) \in [0,1] \times [0,1]$,

$$\begin{aligned} \omega(s,0) &= \omega_p \in E^*_p \\ \nabla_T \omega(s,t) &= 0. \end{aligned}$$

Let $Z_p \in E_p$ and extend by parallel translation first around the loop $\gamma = \Gamma(s,1)$ and then along each curve $\Gamma(s,\cdot)$ back to $\Gamma(s,0) = p$. In particular, for each $(s,t) \in [0,1] \times [0,1]$,

$$\begin{aligned} Z(0,1) &= Z_p \\ Z(1,1) &= P_\gamma Z_p \\ \nabla_S Z(s,1) &= 0 \\ \nabla_T Z(s,t) &= 0. \end{aligned} \tag{1}$$

Observe that, since the curves $\Gamma(0,\cdot)$ and $\Gamma(1,\cdot)$ are constant curves that stay at p, it follows by (1) that

$$\begin{aligned} Z(0,0) &= Z(0,1) = Z_p \\ Z(1,0) &= Z(1,1) = P_\gamma Z_p. \end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \omega(p), P_\gamma Z_p - Z_p \rangle &= \langle \omega(1,0), Z(1,0)\rangle - \langle \omega(0,0), Z(0,0)\rangle \\
&= \int_0^1 \partial_s \langle \omega(s,0), Z(s,0)\rangle \, ds \\
&= \int_0^1 \langle \nabla_S \omega(s,0), Z(s,0)\rangle + \langle \omega(s,0), \nabla_S Z(s,0)\rangle \, dt \\
&= \int_0^1 \langle \omega(s,0), \nabla_S Z(s,0)\rangle \, ds \\
&= \int_0^1 \left[\langle \omega(s,1), \nabla_S Z(s,1)\rangle - \int_0^1 \partial_t \langle \omega(s,t), \nabla_S Z(s,t)\rangle \, dt \right] ds \\
&= -\int_0^1 \int_0^1 \langle \omega(s,t), \nabla_T(\nabla_S Z(s,t))\rangle \, ds \, dt \\
&= \int_0^1 \int_0^1 \langle \omega(s,t), \nabla_S(\nabla_T Z(s,t)) - \nabla_T(\nabla_S Z(s,t)\rangle \, ds \, dt \\
&= \int_0^1 \int_0^1 \langle \omega(s,t), R(S,T) Z(s,t)\rangle \, ds \, dt \\
&= \int_0^1 \int_0^1 \langle \omega(p), \pi_{(s,t)}(\Omega(\partial_s, \partial_t) Z(s,t))\rangle \, ds \, dt \\
&= \int_0^1 \int_0^1 \langle \omega_p, \pi^{-1}_{(s,t)} \Omega(\partial_s, \partial_t) \hat{\pi}_{(s,t)} Z_p \rangle \, ds \, dt \\
&= \left\langle \omega_p, \left[\int_{[0,1]\times[0,1]} \pi^{-1}_{(s,t)} \Omega \hat{\pi}_{(s,t)} \right] Z_p \right\rangle .
\end{aligned}$$

Since this holds for any $\omega_p \in E_p^*$ and $Z_p \in E_p$, the theorem follows. $\square$

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