Notes on Differential Geometry

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August 4, 2018

Chapter 5

Submanifolds of Euclidean space

5.1 Definition of a submanifold

A subset $M^n \subset \mathbb{R}^{n+k}$ is an n-dimensional C^k *submanifold*, if, for any $p \in M$, there exists an open neighborhood N of $0 \in \mathbb{R}^n$ and a C^k embedding $\phi : N \to M \subset \mathbb{R}^{n+k}$ onto a neighborhood of $p \in M$ such that $\phi(0) = p$.

5.2 Notation

Throughout the discussion below, we use the following notation:

$$
\begin{aligned}
M &= \text{an } n\text{-dimensional submanifold in } \mathbb{R}^{n+k} \\
p &= \text{ a point in } M \\
N &= \text{a neighborhood of } 0 \text{ in } \mathbb{R}^n \\
\phi &= \text{a } C^k \text{ embedding of } N \to M \subset \mathbb{R}^{n+k} \text{ such that } \phi(0) = p
\end{aligned}
$$

5.3 Tangent space at a point in a submanifold

The *tangent space* at each $p \in M$ is the space of all possible velocity vectors of C^1 curves passing through p,

$$T_pM = \{\dot{c}(0) \ : \ c : (-\delta, \delta) \to M \text{ is } C^1 \text{ and } c(0) = p\},$$

where $\dot{c}$ denotes the derivative of c. Given an embedding $\phi : N \to M \subset \mathbb{R}^{n+k}$ such that $\phi(0) = p$, a basis of T_pM is given by

$$\partial_i = \left.\frac{d}{dt}\right|_{t=0} \phi(0 + te_i) = \partial_i\phi, \ 1 \le i \le n,$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. Let $dx^1, \dots, dx^n \in T_pM$ denote the dual basis to $\partial_1, \dots, \partial_n$.

5.4 The derivative of a function on a submanifold

Given a C^1 vector-valued function $f : M \to W$ and $v \in T_pM$, the *directional derivative* of f in the direction v at p is defined to be

$$\langle v, df(p) \rangle = \left.\frac{d}{dt}\right|_{t=0} f(c(t)).$$

where $c : (-\delta, \delta)$ satisfies $c(0) = p$, $\dot{c}(0) = v$ and, for each $p \in M$,

$$df(p) = \partial_i(f \circ \phi)(0)\, dx^i \in \mathrm{Hom}(T_pM, W) = T_p^*M \otimes W.$$

is the *differential* of f. In particular, if f is defined on an open neighborhood of $N \subset M$ and $v = v^i \partial_i$, then

$$\begin{aligned} \langle v, df(p) \rangle &= \langle v, \partial_i (f \circ \phi)(0)\, dx^i \rangle \\ &= \partial_i (f \circ \phi)(0) \langle v, dx^i \rangle \\ &= v^i \partial_\alpha f(p) \partial_i \phi^\alpha(0), \end{aligned}$$

where $1 \leq \alpha \leq n + k$.

5.5 The first fundamental form

The first fundamental form at each $p \in M$ is the restriction to $T_p M$ of the standard dot product on $\mathbb{R}^{n+k}$. It is a symmetric 2-form $g : S^2 T_p M \to \mathbb{R}$, given by

$$\begin{aligned} g(p) : S^2 T_p M &\to \mathbb{R} \\ v \otimes w &\mapsto v \cdot w \end{aligned}$$

and is positive definite. It therefore is the Riemannian metric induced by the Euclidean structure, i.e., the dot product, on $\mathbb{R}^{n+k}$.

If $(x^1, \ldots, x^n) \in N$, we can write

$$g = g_{ij} dx^i\, dx^j,$$

where $g_{ij} = g(\partial_i, \partial_j)$. We also define the dual or inverse tensor $g^{-1} = g^{ij} \partial_i\, \partial_j \in S^2 T_* M$, where $g^{ik} g_{kj} = \delta^i_j$ and δ^i_j is the Kronecker delta tensor, i.e., the identity map.

5.6 Intrinsic versus extrinsic geometry

A transformation $R : \mathbb{R}^n \to \mathbb{R}^n$ is a *rigid motion*, if it is a rotation composed with a translation, i.e.

$$Rx = Ax + \tau,$$

where $A \in \mathrm{SO}(n)$ and $\tau \in \mathbb{R}^n$.

Any property of $M \subset \mathbb{R}^{n+k}$ that is invariant under both the local parameterization ϕ and rigid motions of $\mathbb{R}^{n+k}$ is a Euclidean geometric invariant or property. If it, in fact, depends only on the induced metric g, then it is called an *intrinsic geometric property*. If not, it is an *extrinsic property*.

For example, the Riemann curvature of g is an intrinsic geometric invariant.

5.7 The second fundamental form via acceleration

The acceleration of a parameterized curve $c : (-\delta, \delta) \to M$ is simply its second derivative, which can be decomposed into two components, one tangent to M and the other orthogonal to $T_p M$,

$$\ddot{c}(t) = \ddot{c}_T(t) + \ddot{c}_N(t),$$

where

$$\ddot{c}_T = g(\partial_i, \ddot{c})g^{ij}\partial_j$$

and $g^{-1} = g^{ij}\partial_i\,\partial_j \in S^2T_pM$ satisfies

$$g^{ij}g_{jk} = \delta^i{}_k,$$

where $\delta^i{}_k$ is the delta or identity tensor. Note that no local parameterization of M is used to define $\ddot{c}_T$ and $\ddot{c}_N$.

On the other hand, with respect to a local parameterization $\phi : N \to M$, there exists a curve $\chi : (-\delta, \delta) \to N$ such that $c = \phi \circ \chi$. Differentiating this twice,

$$\begin{aligned}\dot{c}(t) &= \partial_i\phi(c(t))\dot{\chi}^i(t)\\ \ddot{c}(t) &= \dot{\chi}^i(t)\dot{\chi}^j(t)\partial^2_{ij}\phi(c(t)) + \ddot{\chi}^i(t)\partial_i\phi(c(t))\\ &= \ddot{c}_T(t) + \ddot{c}_N(t),\end{aligned}$$

where

$$\begin{aligned}\ddot{c}_T &= \ddot{\chi}^i\partial_i\phi + \dot{\chi}^i\dot{\chi}^i(\partial^2_{ij}\phi)_T\\ \ddot{c}_N(t) &= \dot{\chi}^i\dot{\chi}^j(\partial^2_{ij}\phi)_N\\ &= v^iv^j(\partial^2_{ij}\phi)_N\end{aligned}$$

and

$$v = \dot{\chi}^i\partial_i\phi \in T_pM.$$

In particular, the normal component of the acceleration, $\ddot{c}_N$, is a quadratic function of the velocity vector $v = \dot{c}(0)$ and, surprisingly, does not depend on the acceleration of c at all. Since, as observed above, $\ddot{c}_N$ does not depend on the ϕ, it follows that, despite appearances,

$$H = (\partial^2_{ij}\phi)_N\,dx^idx^j \in T_p^\perp M \otimes S^2T_p^*M,$$

also does not depend on the local parameterization ϕ. Since $(\partial^2\phi)_N$ is also invariant under rigid motions, it is a geometric invariant of M. It is called the *second fundamental form* of M.

5.8 The Levi-Civita connection and second fundamental form via the Hessian

Recall that $\partial_1\phi(x), \ldots, \partial_n\phi(x)$ are a basis of $T_{\phi(x)}M$. The Hessian of ϕ can therefore be split into its tangential and normal components,

$$\partial^2_{ij}\phi = (\partial^2\phi)_T + (\partial^2\phi)_N,$$

where the projection onto the tangential component is given by

$$(\partial^2_{ij}\phi)_T = g^{kl}(\partial_k\phi \cdot \partial^2_{ij}\phi)\partial_l\phi.$$

This does not look like a geometric invariant, because it appears to depend on the parameterization ϕ. It also does not look intrinsic. However, there is the following miracle. Since

$$g_{ij} = \partial_i\phi \cdot \partial_j\phi,$$

it follows, by differentiating and writing the same equation three times, cycling through theindices i, j, k,

$$\begin{aligned} \partial_k g_{ij} &= \partial^2_{ki}\phi \cdot \partial_j\phi + \partial_i\phi \cdot \partial^2_{kj}\phi \\ \partial_i g_{jk} &= \partial^2_{ij}\phi \cdot \partial_k\phi + \partial_j\phi \cdot \partial^2_{ik}\phi \\ \partial_i g_{ki} &= \partial^2_{jk}\phi \cdot \partial_i\phi + \partial_k\phi \cdot \partial^2_{ji}\phi. \end{aligned}$$

If we add the last two equations, subtract the first, and divide it all by 2, we get

$$\partial_k\phi \cdot \partial^2_{ij}\phi = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) = g_{kl}\Gamma^l_{ij},$$

where Γ^l_{ij} are the Christoffel symbols of g. Although these are not invariant under the local parameterization of M, they are intrinsic.

5.9 Curvature

We differentiate one more time and again cycle through the indices:

$$\begin{aligned} \partial^2_{kl} g_{ij} &= \partial^3_{kli}\phi \cdot \partial_j\phi + \partial_i\phi \cdot \partial^3_{jkl}\phi + \partial^2_{ik}\phi \cdot \partial^2_{jl}\phi + \partial^2_{il}\phi \cdot \partial^2_{jk}\phi \\ \partial^2_{li} g_{jk} &= \partial^3_{lij}\phi \cdot \partial_k\phi + \partial_j\phi \cdot \partial^3_{kli}\phi + \partial^2_{jl}\phi \cdot \partial^2_{ki}\phi + \partial^2_{ji}\phi \cdot \partial^2_{kl}\phi \\ \partial^2_{ij} g_{kl} &= \partial^3_{ijk}\phi \cdot \partial_l\phi + \partial_k\phi \cdot \partial^3_{lij}\phi + \partial^2_{ki}\phi \cdot \partial^2_{lj}\phi + \partial^2_{kj}\phi \cdot \partial^2_{li}\phi \\ \partial^2_{jk} g_{li} &= \partial^3_{jkl}\phi \cdot \partial_i\phi + \partial_l\phi \cdot \partial^3_{ijk}\phi + \partial^2_{lj}\phi \cdot \partial^2_{ik}\phi + \partial^2_{lk}\phi \cdot \partial^2_{ij}\phi \end{aligned}$$

If we add the first and third equations and subtract the second and fourth, all of the third derivative terms vanish, leaving

$$\frac{1}{2}(\partial^2_{ij} g_{kl} + \partial^2_{kl} g_{ij} - \partial^2_{ik} g_{jl} - \partial^2_{jl} g_{ik}) = \partial^2_{il}\phi \cdot \partial^2_{jk}\phi - \partial^2_{ij}\phi \cdot \partial^2_{kl}\phi$$

It follows that

$$\begin{aligned} (\partial^2_{il}\phi)_N \cdot (\partial^2_{jk}\phi)_N - (\partial^2_{ij}\phi)_N \cdot (\partial^2_{kl}\phi)_N = &\frac{1}{2}(\partial^2_{ij} g_{kl} + \partial^2_{kl} g_{ij} - \partial^2_{ik} g_{jl} - \partial^2_{jl} g_{ik}) \\ &- (\partial^2_{il}\phi)_T \cdot (\partial^2_{jk}\phi)_T - (\partial^2_{ij}\phi)_T \cdot (\partial^2_{kl}\phi)_T \end{aligned}$$

Since the left side consists of tensor products of the second fundamental form, it defines a tensor independent of the parameterization ϕ. Since the right side is intrinsic, so is the left. Therefore, the left side defines an intrinsic geometric invariant,

$$R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where

$$R_{ijkl} = (\partial^2_{ik}\phi)_N \cdot (\partial^2_{jl}\phi)_N - (\partial^2_{il}\phi)_N \cdot (\partial^2_{jk}\phi)_N, \tag{5.1}$$

known as the Riemann curvature. Equation(5.1) is known as the Gauss equations.