

The even Orlicz-Minkowski problem

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Abstract

The classical Minkowski problem lead to the L_p Minkowski problem and now to the Orlicz-Minkowski problem. Existence is demonstrated for the even Orlicz-Minkowski problem. A byproduct is a new approach to the solution of the classical Minkowski problem.

1. Introduction

The Minkowski problem concerns the existence, uniqueness, and stability of convex hypersurfaces whose Gauss curvature (possibly in a generalized sense) is prescribed as a function of the outer unit normals. The Minkowski problem is one of the centerpieces of the classical Brunn-Minkowski theory. The complete solution to the Minkowski problem (for arbitrary “data” — with no smoothness assumptions) goes back exactly three-quarters of a century to the work of Aleksandrov and Fenchel and Jessen (see Schneider [55] for references). Of course, when the Minkowski “data” is discrete then the solution goes back to Minkowski’s work at the turn of the 19th into the 20th Century.

Almost a century after Minkowski unveiled the Brunn-Minkowski theory, beginning largely with [32, 33], an L_p version of the Brunn-Minkowski theory began to emerge. This theory has expanded rapidly. (See e.g. [1 - 5, 7 - 9, 10, 12, 13, 15 - 22, 25 - 38, 40 - 44, 46, 47, 49 - 52, 54, 56, 57 - 59, 61, 62].) The L_p Minkowski problem was a centerpiece of this new L_p -Brunn-Minkowski theory.

The regular L_p Minkowski problem seeks solutions $h : S^{n-1} \rightarrow (0, \infty)$ to

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the PDE

$$h^{1-p} \det(\nabla_{ij}h + e_{ij}h) = g \tag{1}$$

for a given continuous function $g : S^{n-1} \rightarrow \mathbb{R}$ (called the “data”), where the parameter p is a real number and e_{ij} is the standard Riemannian metric on S^{n-1} . The case $p = 1$ is classical with landmark contributions such as Cheng and Yau [6] and Pogorelov [53].

The general L_p Minkowski problem asks: Given a real p , what are the necessary and sufficient conditions on a Borel measure μ on the unit sphere, S^{n-1} , such that there exists a convex body K in \mathbb{R}^n with support function h_K and surface area measure S_K (see Section 1 for definitions) so that

$$h_K^{1-p} dS_K = d\mu. \tag{2}$$

The solutions to the general L_p Minkowski problem (2) may be viewed as weak solutions to the regular L_p Minkowski problem (1). While regularity is a central focus in the fields of partial differential equations and differential geometry, the existence and the uniqueness of weak solutions of the Minkowski problem are of paramount interest to those working in convex geometric and functional analysis. This is because it is precisely these solutions that correspond to support functions of polytopes or to Banach norms (which often have limited smoothness).

The even L_p Minkowski problem seeks solutions under the assumption that the data measure μ is an even Borel measure on S^{n-1} (i.e. assumes the same values on antipodal Borel sets) or the data function g is an even function (i.e. assumes the same values on antipodal points of S^{n-1}). The L_p Minkowski problem with even data is not only natural but of practical interest because its solutions correspond to norms of Banach spaces.

In this paper the “ L_p Minkowski problem” will always refer to the general L_p Minkowski problem (2). The case $p = 1$ of the L_p Minkowski problem (2) is of course the classical Minkowski problem. For $p > 1$, a solution to the even L_p Minkowski problem was given in [32] under the assumption that $p \neq n$. In [39], it was shown that, for $p \neq n$, the L_p Minkowski problem (2) has an equivalent volume-normalized formulation and a solution of the even volume-normalized L_p Minkowski problem was given for all $p > 1$. The regularity of the even L_p Minkowski problem was considered in [34].

In the plane ($n = 2$), the L_p Minkowski problem was treated by Stancu [57, 58, 59], Umanskyi [61], Chen [5], and most recently by Jiang [24].

The L_p Minkowski problem (without the assumption that the data is even) was treated by Guan and Lin [15] and later by Chou and Wang [7].

Hug et al [23] gave an alternate approach to some of the results of Chou and Wang [7].

The solution of the even L_p Minkowski problem was a critical ingredient that allowed the authors of [38] to extend Zhang's affine Sobolev inequality [63] and obtain the L_p affine Sobolev inequality and later enabled Cianchi et al [8] to establish the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities.

Recent work of Haberl and Schuster [19], [20], Haberl, Schuster, and Xiao [21] and the recent work of Ludwig and Reitzner [31], as well as Ludwig [30], shows the need to take the next step in the evolution of the Brunn-Minkowski theory towards an Orlicz-Brunn-Minkowski theory. This will be the third paper in a series that attempts to develop some of the elements of an Orlicz-Brunn-Minkowski theory.

Suppose $\varphi : (0, \infty) \rightarrow (0, \infty)$ is a fixed continuous function. The aim of this paper is to study the even L_φ Minkowski problem: If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1} , then does there exist an origin symmetric convex body K in \mathbb{R}^n such that

$$c\varphi(h_K) dS_K = d\mu,$$

for some positive number c ? We will show that under some mild assumptions on φ , the even L_φ Minkowski problem does have a solution. For example we shall establish:

Theorem. *Suppose $\varphi : (0, \infty) \rightarrow (0, \infty)$ is a continuous decreasing function. If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1} , then there exists an origin symmetric convex body K in \mathbb{R}^n such that*

$$c\varphi(h_K) dS_K = d\mu,$$

where c is a power of the volume of K — specifically, $c = V(K)^{\frac{1}{2n}-1}$.

When φ and the data μ are assumed to be sufficiently smooth, and under certain other restrictions, the existence of solutions to the L_φ Minkowski problem was already demonstrated by Chou and Wang [7]. For applications in analysis (such as in obtaining analytic inequalities of the type presented in [63] and [38]) and for applications in geometry (such as in obtaining affine isoperimetric inequalities), various Banach norms and convex bodies must be constructed. These constructions amount to solving an L_φ Minkowski problem — but usually with minimal restrictions on φ and the measure μ . (Note that a polytopal solution of the L_φ Minkowski problem corresponds to a measure μ whose support is a finite set.) In this paper we will establish

the existence of solutions to the L_φ Minkowski problem with such minimal restrictions on φ and μ .

One interesting feature of our work is that it presents a new approach to the classical Minkowski problem as well as the even L_p Minkowski problem for $p > 0$ and $p \neq n$.

2. Preliminaries

For quick later reference we develop some notation and basic facts about convex bodies. General references for the theory of convex bodies are the excellent books of Gardner [11], Gruber [14], Schneider [55], and Thompson [60].

Our setting will be Euclidean n -space \mathbb{R}^n where $n \geq 2$. The standard inner product of the vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. We write $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ for the boundary of the Euclidean unit ball B in \mathbb{R}^n .

The set of positive continuous functions on the sphere S^{n-1} will be denoted by $C^+(S^{n-1})$. This set will be viewed as equipped with the metric induced by the maximum norm

$$\|f - g\|_\infty = \max_{u \in S^{n-1}} |f(u) - g(u)|.$$

We write $C_e^+(S^{n-1})$ for the subspace of $C^+(S^{n-1})$ consisting of even functions only.

Write V for n -dimensional Lebesgue measure and \mathcal{H}^{n-1} for $(n-1)$ -dimensional Hausdorff measure. The letter μ will be used exclusively to denote a finite Borel measure on S^{n-1} . For such a measure μ , we denote by $|\mu|$ its total mass, i.e. $|\mu| = \mu(S^{n-1})$. The letter c (possibly with subscripts or other distinguishing features) will be used exclusively to denote a positive real number.

A *convex body* is a compact convex subset of \mathbb{R}^n . The set of convex bodies in \mathbb{R}^n containing the origin in their interiors is denoted by \mathcal{K}_o^n . The set of convex bodies in \mathbb{R}^n that are symmetric about the origin will be denoted by \mathcal{K}_e^n .

A compact, convex $K \subset \mathbb{R}^n$ is uniquely determined by its *support function* $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$, where $h_K(x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$. For example, the support function of the line segment \bar{v} joining the points $\pm v \in \mathbb{R}^n$ is given by

$$h_{\bar{v}}(x) = |x \cdot v|, \quad x \in \mathbb{R}^n.$$

We will need the trivial fact that for the support function of the dilate $cK = \{cx : x \in K\}$ of a convex body K we have

$$h_{cK} = ch_K, \quad c > 0. \quad (3)$$

Note that support functions are positively homogeneous of degree one and subadditive. It follows immediately from the definition of support functions that for convex bodies K, L ,

$$K \subset L \iff h_K \leq h_L. \quad (4)$$

Consequently, the support function of a body $K \in \mathcal{K}_o^n$ is bounded from above and below by positive reals.

The set \mathcal{K}_o^n will be viewed as equipped with the Hausdorff metric and thus for a sequence $\{K_i\}$ of bodies in \mathcal{K}_o^n that $\lim_{i \rightarrow \infty} K_i = K \in \mathcal{K}_o^n$ provided that

$$\|h_{K_i} - h_K\|_\infty \rightarrow 0.$$

A boundary point $x \in \partial K$ is said to have $u \in S^{n-1}$ as an outer normal vector provided $x \cdot u = h_K(u)$. A boundary point is said to be *singular* if it has more than one unit normal vector. It is well known (see, e.g., [55]) that the set of singular boundary points of a convex body has \mathcal{H}^{n-1} -measure 0.

For a convex body K and each Borel set $\omega \subset S^{n-1}$, the *reverse spherical image*, $\tau(K, \omega)$, of K at ω is the set of all boundary points of K which have an outer unit normal belonging to the set ω . Associated with each convex body $K \in \mathcal{K}_o^n$ is a Borel measure, S_K , on S^{n-1} called the Aleksandrov-Fenchel-Jessen *surface area measure* of K , defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\tau(K, \omega)), \quad (5)$$

for each Borel set $\omega \subseteq S^{n-1}$. Observe that for the surface area measure of the dilate cK of K we have

$$S_{cK} = c^{n-1}S_K, \quad c > 0. \quad (6)$$

We will also make critical use of the weak continuity of surface area measures; i.e., if $\{K_i\}$ is a sequence of bodies in \mathcal{K}_o^n then

$$\lim_{i \rightarrow \infty} K_i = K \in \mathcal{K}_o^n \implies \lim_{i \rightarrow \infty} S_{K_i} = S_K, \text{ weakly.} \quad (7)$$

The *mixed volume* $V_1(K, L)$ of the convex bodies $K, L \in \mathcal{K}_o^n$ may be defined by

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L dS_K. \quad (8)$$

The fact that

$$V(K) = V_1(K, K),$$

and hence

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h_K dS_K, \quad (9)$$

is of critical importance. The fundamental inequality for mixed volumes is *Minkowski's mixed volume inequality*: For $K, L \in \mathcal{K}_o^n$,

$$V_1(K, L)^n \geq V(K)^{n-1} V(L) \quad (10)$$

with equality if and only K and L are homothetic.

3. Aleksandrov bodies

A function $h \in C^+(S^{n-1})$ defines a family $\{H_u\}_{u \in S^{n-1}}$ of hyperplanes

$$H_u = \{x \in \mathbb{R}^n : x \cdot u = h(u)\}.$$

This family gives rise to concepts such as envelopes in classical differential geometry, generalized envelopes in convex geometric analysis (see, e.g., [48]), and hedgehogs (see, e.g., [45]).

We shall be interested in the intersection of the halfspaces that are associated to h by the family $\{H_u\}_{u \in S^{n-1}}$. This gives rise to the convex body

$$K = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h(u)\}.$$

The body K is called the *Aleksandrov body associated with h* . Note that since h is both positive and continuous its Aleksandrov body, K , must be an element of \mathcal{K}_o^n . The Aleksandrov body associated with h can alternatively be defined as the unique maximal element, with respect to set inclusion, of the set

$$\{Q \in \mathcal{K}_o^n : h_Q \leq h\}.$$

For the Aleksandrov body K associated with h we now examine equality (in S^{n-1}) in the inequality

$$h_K \leq h$$

in some detail. Aleksandrov showed that each element of $\tau(K, \omega_h)$, the reverse spherical image of K of the difference set

$$\omega_h = \{u \in S^{n-1} : h_K(u) < h(u)\},$$

must be a singular boundary point of K . Since the set of singular boundary points of a convex body has \mathcal{H}^{n-1} -measure zero, and since by (5), we know that $S_K(\omega_h) = \mathcal{H}^{n-1}(\tau(K, \omega_h))$, it follows that $S_K(\omega_h) = 0$. Consequently, while $h_K \leq h$,

$$h_K = h, \quad \text{a.e. with respect to } S_K. \quad (11)$$

We will make use of the following three basic properties of Aleksandrov bodies. First, if h is the support function of a convex body $K \in \mathcal{K}_o^n$, then K itself is the Aleksandrov body associated with h . Second, as is easily shown, if h is an even function, then the Aleksandrov body associated with h is origin-symmetric. Third and critical, is *Aleksandrov's convergence lemma*: If the functions $h_i \in C^+(S^{n-1})$ have associated Aleksandrov bodies $K_i \in \mathcal{K}_o^n$, then

$$h_i \rightarrow h \in C^+(S^{n-1}), \text{ uniformly on } S^{n-1} \implies K_i \rightarrow K,$$

where K is the Aleksandrov body associated with h .

The volume $V(h)$ of a function $h \in C^+(S^{n-1})$ is defined as the volume of the Aleksandrov body associated with h . Since the Aleksandrov body associated with the support function h_K of a convex body $K \in \mathcal{K}_o^n$ is the body K itself, we have

$$V(h_K) = V(K). \quad (12)$$

From Aleksandrov's Convergence Lemma and the continuity of volume on \mathcal{K}_o^n we see that

$$V : C^+(S^{n-1}) \rightarrow \mathbb{R} \text{ is continuous.}$$

Let $I \subset \mathbb{R}$ be an interval containing 0 and suppose that $h_t(u) = h(t, u) : I \times S^{n-1} \rightarrow (0, \infty)$ is continuous. For fixed $t \in I$, let

$$K_t = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h(t, u)\}$$

be the Aleksandrov body associated with h_t . The family of bodies $\{K_t\}_{t \in I}$ will be called *the family of Aleksandrov bodies associated with h* . Obviously, from (11) we have

$$h_{K_t} \leq h_t \quad \text{and} \quad h_{K_t} = h_t, \text{ a.e. with respect to } S_{K_t}. \quad (13)$$

for each $t \in I$.

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval containing 0 and some positive number and let $h(t, u) : I \times S^{n-1} \rightarrow (0, \infty)$ be continuous and such that the convergence in*

$$h'_+(0, u) = \lim_{t \rightarrow 0^+} \frac{h(t, u) - h(0, u)}{t} \quad (14)$$

is uniform on S^{n-1} . If $\{K_t\}_{t \in I}$ is the family of Aleksandrov bodies associated with h , then

$$\lim_{t \rightarrow 0^+} \frac{V(K_t) - V(K_0)}{t} = \int_{S^{n-1}} h'_+(0, u) dS_{K_0}(u).$$

Proof. The uniform convergence of (14) implies that $h_t \rightarrow h_0$, uniformly on S^{n-1} . Therefore, the Aleksandrov Convergence Lemma (mentioned above) yields

$$\lim_{t \rightarrow 0^+} K_t = K_0. \quad (15)$$

By (7) we conclude that the S_{K_t} converge weakly to S_{K_0} as $t \rightarrow 0$. Since the measures S_{K_t} are finite, converge weakly to S_{K_0} and since the convergence in

$$\lim_{t \rightarrow 0^+} \frac{h(t, u) - h(0, u)}{t}$$

is uniform on S^{n-1} , we obtain

$$\lim_{t \rightarrow 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_0(u)}{t} dS_{K_t}(u) = \int_{S^{n-1}} h'_+(u, 0) dS_{K_0}(u). \quad (16)$$

Formulas (9) and (13) imply

$$V(K_t) = \frac{1}{n} \int_{S^{n-1}} h_{K_t}(u) dS_{K_t}(u) = \frac{1}{n} \int_{S^{n-1}} h_t(u) dS_{K_t}(u). \quad (17)$$

From (17), (8), and inequality (13) at $t = 0$, we have

$$\begin{aligned} \liminf_{t \rightarrow 0^+} \frac{V(K_t) - V_1(K_t, K_0)}{t} &= \frac{1}{n} \liminf_{t \rightarrow 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_{K_0}(u)}{t} dS_{K_t}(u) \\ &\geq \frac{1}{n} \liminf_{t \rightarrow 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_0(u)}{t} dS_{K_t}(u), \end{aligned}$$

which when combined with (16) gives

$$\liminf_{t \rightarrow 0^+} \frac{V(K_t) - V_1(K_t, K_0)}{t} \geq \frac{1}{n} \int_{S^{n-1}} h'_+(u, 0) dS_{K_0}(u). \quad (18)$$

For the sake of brevity set

$$l = \frac{1}{n} \int_{S^{n-1}} h'_+(u, 0) dS_{K_0}(u).$$

Inequality (18) and Minkowski's mixed volume inequality (10) show

$$l \leq \liminf_{t \rightarrow 0^+} \frac{V(K_t) - V_1(K_t, K_0)}{t} \leq \liminf_{t \rightarrow 0^+} \frac{V(K_t) - V(K_t)^{1-\frac{1}{n}} V(K_0)^{\frac{1}{n}}}{t}.$$

But (15) gives $\lim_{t \rightarrow 0^+} V(K_t) = V(K_0)$ and hence

$$l \leq V(K_0)^{1-\frac{1}{n}} \liminf_{t \rightarrow 0^+} \frac{V(K_t)^{\frac{1}{n}} - V(K_0)^{\frac{1}{n}}}{t}. \quad (19)$$

Now (8), the inequality in (13), and the uniform convergence in (14) give

$$\begin{aligned} \limsup_{t \rightarrow 0^+} \frac{V_1(K_0, K_t) - V(K_0)}{t} &= \frac{1}{n} \limsup_{t \rightarrow 0^+} \int_{S^{n-1}} \frac{h_{K_t}(u) - h_0(u)}{t} dS_{K_0}(u) \\ &\leq \frac{1}{n} \limsup_{t \rightarrow 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_0(u)}{t} dS_{K_0}(u) \\ &= \frac{1}{n} \int_{S^{n-1}} h'_+(u, 0) dS_{K_0}(u) \\ &= l. \end{aligned}$$

This, together with Minkowski's mixed volume inequality (10), yields

$$l \geq \limsup_{t \rightarrow 0^+} \frac{V_1(K_0, K_t) - V(K_0)}{t} \geq \limsup_{t \rightarrow 0^+} \frac{V(K_0)^{1-\frac{1}{n}} V(K_t)^{\frac{1}{n}} - V(K_0)}{t},$$

and hence

$$l \geq V(K_0)^{1-\frac{1}{n}} \limsup_{t \rightarrow 0^+} \frac{V(K_t)^{\frac{1}{n}} - V(K_0)^{\frac{1}{n}}}{t}. \quad (20)$$

Combining (19) and (20) gives

$$l = V(K_0)^{1-\frac{1}{n}} \lim_{t \rightarrow 0^+} \frac{V(K_t)^{\frac{1}{n}} - V(K_0)^{\frac{1}{n}}}{t}. \quad (21)$$

Define a function $g : I \rightarrow \mathbb{R}$ by $g(t) = V(K_t)^{\frac{1}{n}}$. Identity (21) shows that the right derivative of g exists at 0. But this implies that the right derivative of g^n exists at 0 and that

$$\lim_{t \rightarrow 0^+} \frac{g(t)^n - g(0)^n}{t} = n g(0)^{n-1} \lim_{t \rightarrow 0^+} \frac{g(t) - g(0)}{t}.$$

Thus the definition of g and (21) prove

$$\lim_{t \rightarrow 0^+} \frac{V(K_t) - V(K_0)}{t} = nl,$$

which completes the proof of the lemma. \square

Corollary 1. *Let $I \subset \mathbb{R}$ be an interval containing 0 in its interior and let $h(t, u) : I \times S^{n-1} \rightarrow (0, \infty)$ be continuous such that the convergence in*

$$h'(0, u) = \lim_{t \rightarrow 0} \frac{h(t, u) - h(0, u)}{t}$$

is uniform on S^{n-1} . If $\{K_t\}_{t \in I}$ is the family of Aleksandrov bodies associated with h , then

$$\lim_{t \rightarrow 0} \frac{V(K_t) - V(K_0)}{t} = \int_{S^{n-1}} h'(0, u) dS_{K_0}(u).$$

Proof. By Lemma 1, it only remains to show that

$$\lim_{t \rightarrow 0^-} \frac{V(K_t) - V(K_0)}{t} = \int_{S^{n-1}} h'(0, u) dS_{K_0}(u). \quad (22)$$

To that end, define $\tilde{h}(t, u) : -I \times S^{n-1} \rightarrow (0, \infty)$ by $\tilde{h}(t, u) = h(-t, u)$. For the corresponding family $\{\tilde{K}_{-t}\}_{t \in I}$ of Aleksandrov bodies associated with \tilde{h} we have $\tilde{K}_{-t} = K_t$ and $\tilde{K}_0 = K_0$. Thus, by Lemma 1,

$$\lim_{t \rightarrow 0^-} \frac{V(K_t) - V(K_0)}{-t} = \lim_{t \rightarrow 0^+} \frac{V(\tilde{K}_t) - V(\tilde{K}_0)}{t} = \int_{S^{n-1}} \tilde{h}'(0, u) dS_{K_0}(u).$$

Obviously, $\tilde{h}'(0, u) = -h'(0, u)$, which immediately implies (22). \square

The following lemma is a slight variant of a standard result about differentiability under an integral sign. For the sake of completeness, we include a proof.

Lemma 2. *Let $\phi : (0, \infty) \rightarrow (0, \infty)$ be continuously differentiable, $I \subset \mathbb{R}$ be an open interval, and*

$$h : I \times S^{n-1} \rightarrow (0, \infty), \quad (t, u) \mapsto h(t, u)$$

be a continuous function such that the partial derivative $\frac{\partial h}{\partial t}(t, u)$ exists for all $(t, u) \in I \times S^{n-1}$. If $\frac{\partial h}{\partial t}$ is bounded and h is bounded from above and from below by positive numbers, then the function

$$H(t) = \int_{S^{n-1}} (\phi \circ h)(t, u) d\mu(u), \quad t \in I,$$

is differentiable on I and

$$H'(t) = \int_{S^{n-1}} \frac{\partial(\phi \circ h)}{\partial t}(t, u) d\mu(u). \quad (23)$$

Moreover, if $\partial(\phi \circ h)/\partial t$ is continuous with respect to t , then H' is continuous.

Proof. Since ϕ' , the derivative of ϕ , is assumed to be continuous and h is bounded from above and from below by positive numbers, there exists a $c_1 \in (0, \infty)$ such that

$$|(\phi' \circ h)(t, u)| \leq c_1$$

for all $(t, u) \in I \times S^{n-1}$. This, the chain rule, and the assumption that $\frac{\partial h}{\partial t}$ is bounded shows that there exists a $c_2 \in (0, \infty)$ such that

$$\left| \frac{\partial(\phi \circ h)}{\partial t}(t, u) \right| = \left| \left((\phi' \circ h) \cdot \frac{\partial h}{\partial t} \right) (t, u) \right| \leq c_2, \quad (24)$$

for all $(t, u) \in I \times S^{n-1}$.

Let $\{t_i\}$ be a sequence with $\lim_{i \rightarrow \infty} t_i = t \in I$ and $t_i \neq t$ for every $i \in \mathbb{N}$. Set

$$f_i = \frac{(\phi \circ h)(t_i, \cdot) - (\phi \circ h)(t, \cdot)}{t_i - t}, \quad i \in \mathbb{N}.$$

The mean value theorem shows that for each $i \in \mathbb{N}$ and $u \in S^{n-1}$ there exists a $t'_i = t'_i(u) \in I$ such that

$$|f_i(u)| = \left| \frac{\partial(\phi \circ h)}{\partial t}(t'_i, u) \right| \leq c_2,$$

where the last inequality follows from (24). Since the measure μ is assumed to be finite, we therefore proved that all the $|f_i|$ are bounded from above by the same integrable function. By applying the dominated convergence theorem to the sequence $\{f_i\}$ we get the differentiability of H and (23).

The last assertion of the lemma is again an immediate consequence of (24) and the dominated convergence theorem. \square

4. Orlicz norms

Throughout this section, let $\phi : [0, \infty) \rightarrow [0, \infty)$ be strictly increasing, continuously differentiable on $(0, \infty)$ with positive derivative, and satisfy $\lim_{t \rightarrow \infty} \phi(t) = \infty$. Note that under these assumptions, ϕ has an inverse $\phi^{-1} : \phi([0, \infty)) \rightarrow [0, \infty)$ which is continuously differentiable on $\phi((0, \infty))$.

Let μ be a finite Borel measure on the sphere S^{n-1} . For a continuous function $f : S^{n-1} \rightarrow [0, \infty)$, the *Orlicz norm* $\|f\|_\phi$ is defined by

$$\|f\|_\phi = \inf \left\{ \lambda > 0 : \frac{1}{|\mu|} \int_{S^{n-1}} \phi \left(\frac{f}{\lambda} \right) d\mu \leq \phi(1) \right\}. \quad (25)$$

We remark that the Orlicz norm of a function depends not only on ϕ but also on μ although this is not reflected in our notation. Observe that for continuous $f : S^{n-1} \rightarrow [0, \infty)$,

$$\|cf\|_\phi = c\|f\|_\phi, \quad c > 0. \quad (26)$$

In particular we have

$$\|c\|_\phi = c, \quad c > 0. \quad (27)$$

Moreover, it follows immediately from the monotonicity of ϕ that for continuous $f, g : S^{n-1} \rightarrow [0, \infty)$,

$$f \leq g \implies \|f\|_\phi \leq \|g\|_\phi. \quad (28)$$

Lemma 3. *Suppose μ is a finite Borel measure on the sphere and $f : S^{n-1} \rightarrow [0, \infty)$ is a continuous function such that $\mu(\{f > 0\}) > 0$. Then the Orlicz norm $\|f\|_\phi$ of f is positive and*

$$\|f\|_\phi = \lambda_0 \iff \frac{1}{|\mu|} \int_{S^{n-1}} \phi \left(\frac{f}{\lambda_0} \right) d\mu = \phi(1).$$

Proof. Define a function $\psi : (0, \infty) \rightarrow [0, \infty)$, for $\lambda > 0$, by

$$\psi(\lambda) = \frac{1}{|\mu|} \int_{S^{n-1}} \phi \left(\frac{f}{\lambda} \right) d\mu.$$

Since ϕ is strictly increasing and $\mu(\{f > 0\}) > 0$, the function ψ is strictly decreasing. It therefore has an inverse $\psi^{-1} : \psi((0, \infty)) \rightarrow (0, \infty)$.

The dominated convergence theorem and the continuity of ϕ on $(0, \infty)$, show that ψ is continuous, as well.

By the non-negativity of ϕ , Fatou's lemma, and the fact that $\lim_{t \rightarrow \infty} \phi(t) = \infty$ we have

$$\begin{aligned} \liminf_{\lambda \rightarrow 0} \psi(\lambda) &\geq \liminf_{\lambda \rightarrow 0} \frac{1}{|\mu|} \int_{\{f > 0\}} \phi \left(\frac{f}{\lambda} \right) d\mu \\ &\geq \frac{1}{|\mu|} \int_{\{f > 0\}} \liminf_{\lambda \rightarrow 0} \phi \left(\frac{f}{\lambda} \right) d\mu \\ &= \infty, \end{aligned}$$

and hence

$$\lim_{\lambda \rightarrow 0} \psi(\lambda) = \infty. \quad (29)$$

Next, we will show that

$$\phi(1) \in \psi((0, \infty)). \quad (30)$$

Since f is continuous on S^{n-1} , there exists a $c \in (0, \infty)$ with $f(u) < c$ for every $u \in S^{n-1}$. Thus, by the monotonicity of ϕ ,

$$\psi(\lambda) = \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f}{\lambda}\right) d\mu < \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{c}{\lambda}\right) d\mu = \phi(c/\lambda),$$

for every positive λ . In particular, $\psi(2c) < \phi(1/2) < \phi(1)$. From (29) and the continuity of ψ we therefore deduce (30) by the intermediate value theorem.

Finally (30) and the strict monotonicity of ψ prove

$$\|f\|_\phi = \inf \{\lambda > 0 : \psi(\lambda) \leq \phi(1)\} = \psi^{-1}(\phi(1)).$$

This shows that $\|f\|_\phi > 0$ and establishes the desired equivalence. \square

For a continuous $f : S^{n-1} \rightarrow [0, \infty)$ with $\|f\|_\phi > 0$, define

$$\langle f \rangle = \frac{f}{\|f\|_\phi}. \quad (31)$$

From (26) we see that we always have $\|\langle f \rangle\|_\phi = 1$, and now Lemma 3 tells us that if μ is such that $\mu(\{f > 0\}) > 0$, then

$$\frac{1}{|\mu|} \int_{S^{n-1}} \phi \circ \langle f \rangle d\mu = \phi(1). \quad (32)$$

Lemma 4. *Let $\{f_i\}$ be a sequence of nonnegative continuous functions on S^{n-1} with $\mu(\{f_i > 0\}) > 0$ for every i . If*

$$f_i \rightarrow f \quad \text{uniformly on } S^{n-1}$$

and $\mu(\{f > 0\}) > 0$, then

$$\|f_i\|_\phi \rightarrow \|f\|_\phi.$$

Proof. From the uniform convergence, we know that there exists a real $c > 0$ such that $f_i(u) \leq c$ for all i and all $u \in S^{n-1}$. From (27) and (28) we thus obtain

$$0 \leq \|f_i\|_\phi \leq \|c\|_\phi = c$$

for every i . Thus the sequence $\{\|f_i\|_\phi\}$ is bounded. To show that the sequence converges to $\|f\|_\phi$, we prove that every convergent subsequence converges to $\|f\|_\phi$. Denote an arbitrary convergent subsequence of $\{\|f_i\|_\phi\}$ by $\{\|f_i\|_\phi\}$ as well.

To see that $\lim_{i \rightarrow \infty} \|f_i\|_\phi > 0$, suppose the contrary, namely that $\|f_i\|_\phi \rightarrow 0$. Then Lemma 3, the non-negativity of ϕ , Fatou's lemma, and $\lim_{t \rightarrow \infty} \phi(t) = \infty$ would produce the desired contradiction:

$$\begin{aligned} \phi(1) &= \lim_{i \rightarrow \infty} \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu \\ &\geq \liminf_{i \rightarrow \infty} \frac{1}{|\mu|} \int_{\{f>0\}} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu \\ &\geq \frac{1}{|\mu|} \int_{\{f>0\}} \liminf_{i \rightarrow \infty} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu \\ &= \infty. \end{aligned}$$

Thus $\lim_{i \rightarrow \infty} \|f_i\|_\phi > 0$, and from the continuity of ϕ and Lemma 3 we therefore deduce

$$\frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f}{\lim_{i \rightarrow \infty} \|f_i\|_\phi}\right) d\mu = \lim_{i \rightarrow \infty} \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu = \phi(1).$$

Lemma 3 again shows $\lim_{i \rightarrow \infty} \|f_i\|_\phi = \|f\|_\phi$. \square

Lemma 5. *If μ is a finite Borel measure on the sphere S^{n-1} which is not concentrated on a great subsphere of S^{n-1} , then there exists a real $c > 0$ such that $\|h_{\bar{v}}\|_\phi > c$ for every $v \in S^{n-1}$.*

Proof. Note that since μ is not concentrated on a great subsphere of S^{n-1} , we have, for every $v \in S^{n-1}$,

$$\mu(\{h_{\bar{v}} > 0\}) = \mu(S^{n-1} \setminus v^\perp) > 0,$$

where v^\perp denotes the codimension 1 subspace orthogonal to v . Hence Lemma 3 shows that $\|h_{\bar{v}}\|_\phi > 0$ for every unit vector v . In order to establish the assertion of the lemma, it suffices to prove that the function $v \mapsto \|h_{\bar{v}}\|_\phi$ is continuous.

Suppose $v_i \in S^{n-1}$ and $v_i \rightarrow v$. That $h_{\bar{v}_i} \rightarrow h_{\bar{v}}$, uniformly on S^{n-1} is easily seen. Thus $\|h_{\bar{v}_i}\|_\phi$ converges to $\|h_{\bar{v}}\|_\phi$ by Lemma 4. This establishes the desired continuity. \square

5. The even Orlicz-Minkowski problem

Theorem 1. *Suppose $\varphi : (0, \infty) \rightarrow (0, \infty)$ is a continuous decreasing function. If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1} , then for any $0 < \alpha < 1$ there exists an origin symmetric convex body K in \mathbb{R}^n such that*

$$c\varphi(h_K) dS_K = d\mu,$$

where $c = V(K)^{\frac{\alpha}{n}-1}$.

Note that for $\varphi \equiv 1$, Theorem 1 provides a solution to the even Minkowski problem by (6).

Proof. Define the function $\phi : [0, \infty) \rightarrow [0, \infty)$ by

$$\phi(t) = \int_0^t \frac{1}{\varphi(s)} ds.$$

Observe that since φ is decreasing and $0 < \alpha < 1$, it follows that for $c > 0$

$$\lim_{t \rightarrow \infty} \phi(ct)/t^\alpha = \infty, \quad (33)$$

and

$$\lim_{t \rightarrow 0^+} \phi(t)/t^\alpha = 0. \quad (34)$$

Since φ is decreasing, the derivative of ϕ is increasing and therefore the function ϕ is convex.

Furthermore, for $c > 0$ let $\psi_c : (0, \infty) \rightarrow \mathbb{R}$ be defined as

$$\psi_c(t) = \left(\frac{n}{\alpha} V(B)^{\frac{\alpha}{n}} - \frac{|\mu|\phi(ct)}{t^\alpha} \right) t^\alpha.$$

From (33) we conclude that $\psi_c(t)$ converges to $-\infty$ as t tends to infinity. In particular, for each c there exists a real $r_c > 0$ such that

$$t > r_c \implies \psi_c(t) \leq 0. \quad (35)$$

Motivated by the work of Chou and Wang [7], we define the functional $\Phi : C_e^+(S^{n-1}) \rightarrow \mathbb{R}$ for $f \in C_e^+(S^{n-1})$, by

$$\Phi(f) = \frac{n}{\alpha} V(f)^{\frac{\alpha}{n}} - \int_{S^{n-1}} \phi \circ f d\mu.$$

Since, as seen in Section 3, the volume $V : C^+(S^{n-1}) \rightarrow \mathbb{R}$ is continuous, Φ is continuous as well.

Note that by (12), for every $r > 0$,

$$\Phi(h_{rB}) = \frac{n}{\alpha} r^\alpha V(B)^{\frac{\alpha}{n}} - \phi(r)|\mu| = \left(\frac{n}{\alpha} V(B)^{\frac{\alpha}{n}} - \frac{|\mu|\phi(r)}{r^\alpha} \right) r^\alpha.$$

From this and (34) we see that $\Phi(h_{rB})$ is positive for small positive r . Hence

$$\exists K \in \mathcal{K}_e^n \quad \text{such that} \quad \Phi(h_K) > 0. \quad (36)$$

We are searching for a function at which Φ attains a maximum. As will be seen, the search can be restricted to support functions of origin symmetric convex bodies. Indeed, recall from Section 3 that the Aleksandrov body K associated with a given function $h \in C_e^+(S^{n-1})$ is origin symmetric and has a support function h_K which satisfies $0 < h_K \leq h$. Since ϕ is increasing and $V(h) = V(h_K)$ by (12) we deduce $\Phi(h) \leq \Phi(h_K)$.

Next, we will show that the search for a function at which Φ attains a maximum can be further restricted to support functions of origin symmetric bodies contained in some ball of fixed radius. To this end, first note that the continuous function

$$v \mapsto \int_{S^{n-1}} h_{\bar{v}} d\mu, \quad v \in S^{n-1},$$

is positive since μ is not concentrated on a great subsphere and thus, there exists a $c \in (0, \infty)$ such that

$$\frac{1}{|\mu|} \int_{S^{n-1}} h_{\bar{v}} d\mu \geq c, \quad (37)$$

for every $v \in S^{n-1}$. Let $K \in \mathcal{K}_e^n$ and choose $v_K \in S^{n-1}$ such that for a suitable $r_K > 0$ the point $r_K v_K$ is an element of K with maximal distance from the origin. Since K is origin symmetric, the line segment with endpoints $\pm r_K v_K$ is contained in K . From (3) and (4) we deduce $r_K h_{\bar{v}_K} \leq h_K$. The monotonicity of ϕ , Jensen's inequality, and (37) therefore yield

$$\begin{aligned} \int_{S^{n-1}} \phi(h_K) d\mu &\geq \int_{S^{n-1}} \phi(r_K h_{\bar{v}_K}) d\mu \\ &\geq |\mu| \phi \left(\frac{1}{|\mu|} \int_{S^{n-1}} r_K h_{\bar{v}_K} d\mu \right) \\ &\geq |\mu| \phi(cr_K). \end{aligned}$$

Now (12), the fact that $K \subset r_K B$, and the last inequality show that

$$\begin{aligned}
\Phi(h_K) &= \frac{n}{\alpha} V(K)^{\frac{\alpha}{n}} - \int_{S^{n-1}} \phi(h_K) d\mu \\
&\leq \frac{n}{\alpha} r_K^\alpha V(B)^{\frac{\alpha}{n}} - |\mu| \phi(cr_K) \\
&= \left(\frac{n}{\alpha} V(B)^{\frac{\alpha}{n}} - \frac{|\mu| \phi(cr_K)}{r_K^\alpha} \right) r_K^\alpha \\
&= \psi_c(r_K).
\end{aligned}$$

From (35) we therefore conclude that there exists a real $r = r_c > 0$ such that

$$r_K > r \implies \Phi(h_K) \leq 0. \quad (38)$$

It follows from (36) and (38) that in order to find a maximum of the functional Φ on $C_e^+(S^{n-1})$, it is sufficient to search among support functions of members of the set

$$\mathcal{F} = \{K \in \mathcal{K}_e^n : K \subset rB\}.$$

Let $\{K_i\}$, $K_i \in \mathcal{F}$, be a maximizing sequence for Φ , i.e.

$$\lim_{i \rightarrow \infty} \Phi(h_{K_i}) = \sup\{\Phi(h_K) : K \in \mathcal{F}\}.$$

Obviously, the sequence $\{K_i\}$ is bounded. Blaschke's selection theorem (see, e.g., [55]) guarantees the existence of a convergent subsequence, which we also denote by $\{K_i\}$, with $\lim_{i \rightarrow \infty} K_i = K_0$. Since the $K_i \in \mathcal{K}_e^n$, the body K_0 is origin symmetric. Moreover, the continuity of volume, (12), and the positivity of $\lim_{i \rightarrow \infty} \Phi(h_{K_i})$ yield

$$\frac{n}{\alpha} V(K_0)^{\frac{\alpha}{n}} = \lim_{i \rightarrow \infty} \frac{n}{\alpha} V(K_i)^{\frac{\alpha}{n}} = \lim_{i \rightarrow \infty} \frac{n}{\alpha} V(h_{K_i})^{\frac{\alpha}{n}} \geq \lim_{i \rightarrow \infty} \Phi(h_{K_i}) > 0.$$

Consequently, the body K_0 has non-empty interior and thus $K_0 \in \mathcal{K}_e^n \cap \mathcal{K}_o^n$. The continuity of Φ now implies, for every $f \in C_e^+(S^{n-1})$,

$$\Phi(f) \leq \Phi(h_{K_0}).$$

Suppose $f \in C_e^+(S^{n-1})$. For sufficiently small $\delta > 0$ we can define a function, $h : (-\delta, \delta) \times S^{n-1} \rightarrow (0, \infty)$, which is bounded from above and below by positive reals, by

$$h_t(u) := h(t, u) = h_{K_0}(u) + tf(u).$$

By Lemma 2 and Corollary 1, the function $t \mapsto \Phi \circ h_t$ is differentiable at 0. Since K_0 is a maximizer of the functional Φ and $h_0 = h_{K_0}$ we have

$$\left. \frac{d}{dt}(\Phi \circ h_t) \right|_{t=0} = 0.$$

Calculating this derivative, Lemma 2 and Corollary 1 give

$$V(K_0)^{\frac{\alpha}{n}-1} \int_{S^{n-1}} f dS_{K_0} - \int_{S^{n-1}} \frac{1}{\varphi \circ h_{K_0}} f d\mu = 0.$$

Since this holds for all positive even continuous functions f on S^{n-1} , it holds for all even continuous f on S^{n-1} and hence

$$V(K_0)^{\frac{\alpha}{n}-1} \varphi(h_{K_0}) dS_{K_0} = d\mu.$$

□

We now establish the following variant of our first theorem.

Theorem 2. *Suppose $\varphi : (0, \infty) \rightarrow (0, \infty)$ is a continuous function such that $\int_0^t 1/\varphi(s) ds$ exists for every positive t and is unbounded as t tends to ∞ . If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1} , then there exists an origin symmetric convex body K in \mathbb{R}^n such that*

$$c\varphi(h_K) dS_K = d\mu.$$

for some $c \in (0, \infty)$.

Let φ be as in Theorem 2 and define $\phi : [0, \infty) \rightarrow [0, \infty)$ by

$$\phi(t) = \int_0^t \frac{1}{\varphi(s)} ds, \quad \text{for } t > 0, \quad \text{and} \quad \phi(0) = \lim_{t \rightarrow 0^+} \phi(t).$$

The function ϕ is strictly increasing and continuously differentiable on $(0, \infty)$, and $\phi' > 0$. Thus ϕ has an inverse $\phi^{-1} : \phi([0, \infty)) \rightarrow [0, \infty)$ which is continuously differentiable on $\phi((0, \infty))$. Observe that $\lim_{t \rightarrow \infty} \phi(t) = \infty$.

Suppose that $f \in C^+(S^{n-1})$ and $K \in \mathcal{K}_o^n$. Recall that the function ϕ is strictly increasing. There exists $\delta > 0$ so that

$$\phi(1) - tf(u) \in \phi((\frac{1}{2}, 2))$$

for all $(t, u) \in (-\delta, \delta) \times S^{n-1}$. Define a function $\tilde{h} : (-\delta, \delta) \times S^{n-1} \rightarrow (0, \infty)$, at (t, u) , by

$$\tilde{h}(t, u) = \frac{h_K(u)}{\phi^{-1}(\phi(1) - tf(u))}. \quad (39)$$

Clearly, \tilde{h} is bounded from above and below by positive reals (dependent on f and K). From (27) and (28) we therefore have that also $\|\tilde{h}_t\|_\phi$ is bounded from above and below by positive reals.

The following lemma follows directly from the definitions.

Lemma 6. *The function \tilde{h} is continuous and the partial derivative $\frac{\partial \tilde{h}}{\partial t}(t, u)$ exists on $(-\delta, \delta) \times S^{n-1}$ and is given by*

$$\frac{\partial \tilde{h}}{\partial t}(t, u) = \frac{h_K(u)f(u)(\phi^{-1})'(\phi(1) - tf(u))}{[\phi^{-1}(\phi(1) - tf(u))]^2}. \quad (40)$$

Moreover, it is continuous and bounded from above and below by positive reals.

Lemma 7. *The function $t \rightarrow V(\tilde{h}_t)$ is differentiable at 0.*

Proof. By Lemma 6, the function \tilde{h} is continuous. So in order to apply Corollary 1 it remains to show that the limit

$$\lim_{t \rightarrow 0} \frac{\tilde{h}_t(u) - \tilde{h}_0(u)}{t} \quad (41)$$

exists uniformly on S^{n-1} . Let $0 < \delta_1 < \delta$. By the mean value theorem, for every $t \in [-\delta_1, \delta_1]$ we can find a $t_1 \in [-t, t]$ such that

$$\left| \frac{\tilde{h}_t(u) - \tilde{h}_0(u)}{t} - \tilde{h}'_0(u) \right| = \left| \frac{\partial \tilde{h}}{\partial t}(t_1, u) - \frac{\partial \tilde{h}}{\partial t}(0, u) \right|.$$

By compactness, Lemma 6 shows that the partial derivative $\frac{\partial \tilde{h}}{\partial t}(t, u)$ is uniformly continuous on $[-\delta_1, \delta_1] \times S^{n-1}$. Hence, the limit (41) does indeed exist uniformly on S^{n-1} . \square

Lemma 8. *The function $t \rightarrow \|\tilde{h}_t\|_\phi$ is differentiable on $(-\delta, \delta)$ with bounded derivative.*

Proof. As mentioned after the definition of \tilde{h} , there exist $c_1, c_2 \in (0, \infty)$ such that

$$c_1 < \tilde{h}_t(u) < c_2, \quad \text{for all } (t, u) \in (-\delta, \delta) \times S^{n-1}.$$

Define a function $G : (-\delta, \delta) \times (c_1, c_2) \rightarrow \mathbb{R}$ at (t, λ) by

$$G(t, \lambda) = \frac{1}{|\mu|} \int_{S^{n-1}} \phi \left(\frac{\tilde{h}_t}{\lambda} \right) d\mu - \phi(1).$$

Furthermore, we define, for fixed $\lambda \in (c_1, c_2)$, a function $h_1 : (-\delta, \delta) \times S^{n-1} \rightarrow (0, \infty)$ by

$$h_1(t, u) = \tilde{h}(t, u)/\lambda,$$

and, for fixed $t \in (-\delta, \delta)$, a function $h_2 : (c_1, c_2) \times S^{n-1} \rightarrow (0, \infty)$ by

$$h_2(\lambda, u) = \tilde{h}(t, u)/\lambda.$$

Clearly, the functions h_1 and h_2 are bounded from above and below by positive reals. Moreover, by Lemma 6, the derivatives $\frac{\partial h_1}{\partial t}$ and $\frac{\partial h_2}{\partial \lambda}$ exist, are continuous, and bounded on their domains. Thus, by applying Lemma 2 to h_1 and h_2 respectively, the partial derivatives $\partial G/\partial t$, $\partial G/\partial \lambda$ exist. Since $\partial(\phi \circ h_1)/\partial t$ and $\partial(\phi \circ h_2)/\partial \lambda$ are continuous by Lemma 6, another application of Lemma 2 shows that $\partial G/\partial t$ and $\partial G/\partial \lambda$ are in fact continuous. Since we are allowed by Lemma 2 to interchange differentiation and integration, and elementary calculation shows that $\partial G/\partial t$ and $-\partial G/\partial \lambda$ are bounded from above and below by positive reals. In particular, $\partial G/\partial \lambda$ is always nonzero.

Let $t \in (-\delta, \delta)$. For sufficiently small ε , Lemma 3 and the meanvalue theorem give

$$\begin{aligned} 0 &= G(t + \varepsilon, \|\tilde{h}_{t+\varepsilon}\|_\phi) - G(t, \|\tilde{h}_t\|_\phi) \\ &= \varepsilon \frac{\partial G}{\partial t}(x_\varepsilon) + (\|\tilde{h}_{t+\varepsilon}\|_\phi - \|\tilde{h}_t\|_\phi) \frac{\partial G}{\partial \lambda}(x_\varepsilon), \end{aligned} \quad (42)$$

where x_ε is a point on the line segment joining the points $(t, \|\tilde{h}_t\|_\phi)$ and $(t + \varepsilon, \|\tilde{h}_{t+\varepsilon}\|_\phi)$.

Note that the continuity of \tilde{h} and compactness, \tilde{h} is uniformly continuous on $[-\delta_1, \delta_1] \times S^{n-1}$ for arbitrary $0 < \delta_1 < \delta$. Thus the convergence in $\lim_{\varepsilon \rightarrow 0} \tilde{h}_{t+\varepsilon} = \tilde{h}_t$ is uniform and hence, by Lemma 4, $\lim_{\varepsilon \rightarrow 0} \|\tilde{h}_{t+\varepsilon}\|_\phi = \|\tilde{h}_t\|_\phi$. Hence x_ε converges to $(t, \|\tilde{h}_t\|_\phi)$ as ε tends to zero.

Since, as shown above, the partial derivatives of G are continuous and $\partial G/\partial \lambda$ is always nonzero, we conclude from (42) that $\|\tilde{h}_t\|_\phi$ is differentiable with

$$\frac{d\|\tilde{h}_t\|_\phi}{dt} = - \left[\frac{\partial G}{\partial t}(t, \|\tilde{h}_t\|_\phi) \right] / \left[\frac{\partial G}{\partial \lambda}(t, \|\tilde{h}_t\|_\phi) \right].$$

The fact that $\partial G/\partial t$ and $-\partial G/\partial \lambda$ are bounded from above and below by positive reals, finally shows that $d\|\tilde{h}_t\|_\phi/dt$ is bounded. \square

Now, we are in a position to establish some differentiability properties of functions involving \tilde{h} . Recall that

$$\langle \tilde{h}_t \rangle = \frac{\tilde{h}_t}{\|\tilde{h}_t\|_\phi}.$$

Lemma 9. *The functions*

$$t \mapsto \frac{\|\tilde{h}_t\|_\phi}{V(\tilde{h}_t)^{\frac{1}{n}}} \quad \text{and} \quad t \mapsto \int_{S^{n-1}} \phi(\langle \tilde{h}_t \rangle) d\mu$$

are differentiable at 0 with derivatives

$$\frac{1}{V(\tilde{h}_0)^{\frac{1}{n}}} \left(\left. \frac{d}{dt} \|\tilde{h}_t\|_\phi \right|_{t=0} - \frac{\|\tilde{h}_0\|_\phi}{nV(\tilde{h}_0)} \int_{S^{n-1}} \frac{\tilde{h}_0 f}{\phi'(1)} dS_K \right) \quad (43)$$

and

$$\frac{1}{\|\tilde{h}_0\|_\phi} \int_{S^{n-1}} \frac{1}{\varphi(\langle \tilde{h}_0 \rangle)} \left(\frac{\tilde{h}_0 f}{\phi'(1)} - \langle \tilde{h}_0 \rangle \left. \frac{d}{dt} \|\tilde{h}_t\|_\phi \right|_{t=0} \right) d\mu, \quad (44)$$

respectively.

Proof. The differentiability at 0 of

$$t \mapsto \frac{\|\tilde{h}_t\|_\phi}{V(\tilde{h}_t)^{\frac{1}{n}}}$$

is an immediate consequence of Lemmas 7 and 8. Formula (43) for the respective derivative follows directly from (40) and Corollary 1.

Since \tilde{h} and $\|\tilde{h}_t\|_\phi$ are bounded from below and above by positive reals, so is $\langle \tilde{h}_t \rangle$. From Lemmas 6 and 8 we infer that $\langle \tilde{h}_t \rangle$ is differentiable with respect to t . An elementary calculation shows that

$$\frac{\partial \langle \tilde{h}_t \rangle}{\partial t} = \frac{1}{\|\tilde{h}_t\|_\phi} \left(\frac{d\tilde{h}_t}{dt} - \langle \tilde{h}_t \rangle \frac{d\|\tilde{h}_t\|_\phi}{dt} \right). \quad (45)$$

Since $\|\tilde{h}_t\|_\phi$ is bounded from below and above by positive reals and $\langle \tilde{h}_t \rangle$, $d\tilde{h}_t/dt$, $d\|\tilde{h}_t\|_\phi/dt$ are bounded, we therefore conclude that $\partial \langle \tilde{h}_t \rangle / \partial t$ is bounded as well. Lemma 2 proves the differentiability of

$$t \mapsto \int_{S^{n-1}} \phi(\langle \tilde{h}_t \rangle) d\mu$$

at 0 and, together with (45) and (40), formula (44). \square

We are now in a position to establish Theorem 2.

Proof. Define the functional $\Phi : C_e^+(S^{n-1}) \rightarrow \mathbb{R}$ for $f \in C_e^+(S^{n-1})$, by

$$\Phi(f) = \frac{\|f\|_\phi}{V(f)^{\frac{1}{n}}}.$$

Lemma 4, together with the fact that $V : C_e^+(S^{n-1}) \rightarrow \mathbb{R}$ is continuous, shows that Φ is continuous as well.

We are searching for a function at which Φ attains a minimum. As will be seen, the search can be restricted to support functions of origin symmetric convex bodies. Indeed, recall that the Aleksandrov body K associated with a given function $h \in C_e^+(S^{n-1})$ is origin symmetric and has a support function h_K which satisfies $0 < h_K \leq h$. The fact that $V(h) = V(h_K)$ together with (28) shows that $\Phi(h_K) \leq \Phi(h)$.

Since Φ is positively homogeneous of degree 0, the search can be further restricted to support functions of convex bodies with unit volume. Let c_1 denote the value of Φ evaluated at the centered Euclidean ball with unit volume. It follows that in order to find a minimum of the functional Φ on $C_e^+(S^{n-1})$, it is sufficient to search among the support functions of the members of the set

$$\mathcal{F} = \{K \in \mathcal{K}_e^n : \Phi(h_K) \leq c_1 \text{ and } V(K) = 1\}.$$

Let $\{K_i\}$, $K_i \in \mathcal{F}$, be a minimizing sequence for Φ , i.e.

$$\lim_{i \rightarrow \infty} \Phi(h_{K_i}) = \inf\{\Phi(h_K) : K \in \mathcal{F}\}.$$

We claim that the sequence $\{K_i\}$ is bounded. For each i , let $v_i \in S^{n-1}$ be chosen such that for suitable $r_i > 0$ the points $r_i v_i$ are elements of K_i with maximal distance from the origin. Since each K_i is origin symmetric, the segments with endpoints $\pm r_i v_i$ are contained in K_i . From (3) and (4) we deduce $r_i h_{\bar{v}_i} \leq h_{K_i}$. Hence (26), (28), and the fact that the K_i belong to \mathcal{F} imply

$$r_i \|h_{\bar{v}_i}\|_\phi = \|r_i h_{\bar{v}_i}\|_\phi \leq \|h_{K_i}\|_\phi = \Phi(h_{K_i}) \leq c_1. \quad (46)$$

Moreover, by Lemma 5, there exists a $c_2 \in (0, \infty)$ such that

$$c_2 \leq \|h_{\bar{v}_i}\|_\phi \quad (47)$$

for all i . Combining (46) and (47) we see that the r_i are bounded from above and hence the sequence $\{K_i\}$ is bounded. Now Blaschke's selection theorem

guarantees the existence of a convergent subsequence of $\{K_i\}$, which we also denote by $\{K_i\}$, with $\lim_{i \rightarrow \infty} K_i = K_0$. Clearly, the body K_0 is again an origin symmetric convex body. Since it has unit volume, it is in fact contained in \mathcal{K}_o^n . The continuity of Φ now implies that $K_0 \in \mathcal{F}$ and thus that

$$\Phi(h_{K_0}) \leq \Phi(f)$$

for every $f \in C_e^+(S^{n-1})$.

For fixed $f \in C_e^+(S^{n-1})$, let $\tilde{h} : (-\delta, \delta) \times S^{n-1} \rightarrow (0, \infty)$ be defined as in (39), i.e.

$$\tilde{h}(t, u) = \frac{h_{K_0}(u)}{\phi^{-1}(\phi(1) - tf(u))}.$$

Lemma 9 shows that the function $t \mapsto \Phi \circ \tilde{h}_t$ is differentiable at 0. Since h_{K_0} is a minimizer of the functional Φ and $\tilde{h}_0 = h_{K_0}$ we have

$$\left. \frac{d}{dt} (\Phi \circ \tilde{h}_t) \right|_{t=0} = 0.$$

The expression for the above derivative given in (43) implies

$$\left. \frac{d}{dt} \|\tilde{h}_t\|_\phi \right|_{t=0} = \frac{\|\tilde{h}_0\|_\phi}{nV(\tilde{h}_0)} \int_{S^{n-1}} \frac{\tilde{h}_0 f}{\phi'(1)} dS_{K_0}. \quad (48)$$

For each t such that $|t| < \delta$, we have from (32)

$$\frac{1}{|\mu|} \int_{S^{n-1}} \phi \circ \langle \tilde{h}_t \rangle d\mu = \phi(1).$$

If we differentiate this equation at $t = 0$ and use (44) then we get

$$\int_{S^{n-1}} \frac{1}{\varphi \circ \langle \tilde{h}_0 \rangle} \left(\frac{\tilde{h}_0 f}{\phi'(1)} - \langle \tilde{h}_0 \rangle \left. \frac{d}{dt} \|\tilde{h}_t\|_\phi \right|_{t=0} \right) d\mu = 0.$$

By inserting (48) in the above equation and using (12) as well as the fact that $\tilde{h}_0 = h_{K_0}$ we arrive at

$$\frac{1}{nV(K_0)} \int_{S^{n-1}} h_{K_0} f dS_{K_0} \int_{S^{n-1}} \frac{h_{K_0}}{\varphi \circ \langle \tilde{h}_0 \rangle} d\mu = \int_{S^{n-1}} \frac{h_{K_0} f}{\varphi \circ \langle \tilde{h}_0 \rangle} d\mu.$$

Note that by (3), (6), and the homogeneity of volume (of degree n), this equation remains unchanged if we replace K_0 by a dilate of K_0 . In particular, if we choose a dilate K_1 of K_0 such that

$$\int_{S^{n-1}} \frac{h_{K_1}}{\varphi \circ \langle \tilde{h}_0 \rangle} d\mu = n,$$

then we obtain

$$\frac{1}{V(K_1)} \int_{S^{n-1}} h_{K_1} f dS_{K_1} = \int_{S^{n-1}} \frac{h_{K_1} f}{\varphi \circ \langle \tilde{h}_0 \rangle} d\mu.$$

Since this holds for all positive even continuous f on S^{n-1} , it holds for all even continuous f on S^{n-1} , and hence

$$\frac{1}{V(K_1)} dS_{K_1} = \frac{1}{\varphi \circ \langle \tilde{h}_0 \rangle} d\mu.$$

Note that by the definition of $\langle \cdot \rangle$, the fact that $\tilde{h}_0 = h_{K_0}$, (3), and (26) we have

$$\langle \tilde{h}_0 \rangle = \frac{\tilde{h}_0}{\|\tilde{h}_0\|_\phi} = \frac{h_{K_0}}{\|h_{K_0}\|_\phi} = \frac{h_{K_1}}{\|h_{K_1}\|_\phi},$$

and thus

$$\frac{1}{V(K_1)} \varphi \left(\frac{h_{K_1}}{\|h_{K_1}\|_\phi} \right) dS_{K_1} = d\mu.$$

Finally, set $K_2 = \|h_{K_1}\|_\phi^{-1} K_1$. Then (6) and the homogeneity of volume yield

$$c\varphi(h_{K_2}) dS_{K_2} = d\mu$$

where $c = (\|h_{K_1}\|_\phi V(K_2))^{-1}$. □

As a consequence of Theorem 2, by taking $\varphi(t) = t^{1-p}$ for $p > 0$, we obtain the solution to the even L_p Minkowski problem for positive $p \neq n$.

Corollary 2. *If μ is an even finite Borel measure on the sphere S^{n-1} which is not concentrated on a great subsphere of S^{n-1} , then for each $0 < p \neq n$ there exists an origin symmetric convex body K in \mathbb{R}^n such that*

$$h_K^{1-p} dS_K = d\mu.$$

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