ORLICZ PROJECTION BODIES

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As Schneider [50] observes, the classical Brunn-Minkowski theory had its origin at the turn of the 19th into the 20th century, when Minkowski joined a method of combining convex bodies (which became known as *Minkowski addition*) with that of ordinary volume. One of the core concepts that Minkowski introduced within the Brunn-Minkowski theory is that of *projection body* (precise definitions to follow). Four decades ago, in a highly influential paper, Bolker [1] illustrated how Minkowski's *projection operator*, its range (called the class of *zonoids*), and its polar were in fact objects of independent investigation in a number of disciplines.

Within the Brunn-Minkowski theory, the two classical inequalities which connect the volume of a convex body with that of its *polar projection body* are the Petty and Zhang projection inequalities. In retrospect, it is interesting to observe that these inequalities did not emerge out of Blaschke's school, and that it took seven decades from Minkowski's discovery of projection bodies, for the *Petty projection inequality* to appear (see e.g., the books by Gardner [11], Leichweiss [21], Schneider [50], and Thompson [53] for reference). It took yet another two decades for the *Zhang projection inequality* to be discovered. Establishing the analogs of the Petty and Zhang projection inequalities for the projection operator (as opposed to the polar projection operator) are major open problems within the field of convex geometric analysis.

Unlike the classical isoperimetric inequality, the Petty and Zhang projection inequalities are affine isoperimetric inequalities in that they are inequalities between a pair of geometric functionals whose ratio is invariant under affine transformations. The Petty projection inequality is not only stronger than (i.e., directly implies) the classical isoperimetric inequality, but it can be viewed as an optimal isoperimetric inequality. In the same manner that the classical isoperimetric inequality leads to (in fact is equivalent to) the Sobolev inequality, the Petty projection inequality has lead to *Zhang's affine Sobolev inequality* [56] that is stronger than (directly implies) the classical Sobolev inequality and yet is independent of any underlying Euclidean structure.

In the early 1960's, Firey (see e.g. Schneider [50]) introduced an L_p-extension of Minkowski's addition (now known as *Firey-Minkowski L_p-addition*) of convex bodies. In the mid 1990s, it was shown in [30, 31], that when Firey-Minkowski L_p-addition is combined with volume the result is an embryonic L_p-Brunn-Minkowski theory. This theory has expanded rapidly. (See e.g. [2–8, 14–19, 22–28, 30–43, 49, 51, 52, 54, 55].)

An early achievement of the new L_p Brunn-Minkowski theory was the discovery of L_p-analogs of projection bodies and of the Petty projection inequality [33], with an alternate approach to establishing this inequality presented by Campi and

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Gronchi in [2]. These new inequalities have found applications in the field of analytic inequalities where they led to affine L_p Sobolev inequalities [36] and ultimately to affine Moser-Trudinger and affine Morrey-Sobolev inequalities [7].

Work of Ludwig [25] (see also [22]) showed that the known L_p extension of the projection operator considered in [33] is only one of a family of natural L_p extensions of their classical counterpart. Using this insight, Haberl and Schuster [17] (see also [18]) obtained so called "asymmetric" L_p-analogs of the Petty projection inequality. For bodies that are not symmetric about the origin, the inequalities of Haberl and Schuster are stronger than the original L_p Petty projection inequality. The operators considered by Haberl and Schuster appear to be ideally suited for non-symmetric bodies. This can be seen most clearly by looking at the L_p analog of the classical *Blaschke-Santaló inequality* presented in [41]. For origin symmetric bodies, this L_p extension does recover the original Blaschke-Santaló inequality as $p \to \infty$. However, for arbitrary bodies only the Haberl-Schuster version does so.

The above cited work of Haberl and Schuster and the recent work of Ludwig and Reitzner [28], as well as Ludwig [27], makes it apparent that the time is ripe for the next step in the evolution of the Brunn-Minkowski theory towards the Orlicz-Brunn-Minkowski theory. This will be the first paper in a series that attempts to develop some of the elements of an Orlicz-Brunn-Minkowski theory.

It is the aim of this paper to define Orlicz projection bodies and establish the Orlicz analog of the classical Petty projection inequality. Obviously, the new inequality has all its predecessors (including the Haberl-Schuster version) as special cases.

Another classical affine isoperimetric inequality is the *Busemann-Petty centroid inequality*. This is an inequality between the volume of a convex body and that of its *centroid body*. The centroid body is a concept that goes back at least to Dupin. Another early achievement of the L_p Brunn-Minkowski theory was the L_p-extension of this classical concept and the establishment of the L_p-analog of the Busemann-Petty centroid inequality [33], [2]. The L_p extensions of the centroid operator quickly became an object of interest in asymptotic geometric analysis (see e.g. [9], [10], [20], [44], [45], [46], [47]) and even the theory of stable distributions (see [43]).

It was shown in [29] that once the Busemann-Petty centroid inequality has been established, the Petty projection inequality could be obtained as an almost effortless consequence. In addition, it was shown in [29] that also the reverse is the case: the Busemann-Petty centroid inequality could also be obtained easily from the Petty projection inequality. As shown in [33], this turned out to also be the relation between the L_p Petty projection inequality and the L_p Busemann-Petty centroid inequalities: only one of these two inequalities needs to be established and then the other could be quickly derived as a consequence. It appears that this might not be the case for the Orlicz analogues of these classical inequalities. Neither the Orlicz Petty projection inequality nor the Orlicz Busemann-Petty centroid inequality appears to lead to the other in some manner discernable to the authors. Therefore, the topic of Orlicz centroid bodies will be treated in a subsequent work.

We consider convex $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$. This means that ϕ must be decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. We will assume throughout that one of these is happening strictly so; i.e., ϕ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$.

Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and that has volume $|K|$. The Orlicz projection body $\Pi_\phi K$ of K is defined as the body whose support function (see Section 1 for definitions) is given by

$$h_{\Pi_\phi K}(x) = \inf\left\{\lambda > 0 : \int_{\partial K} \phi\left(\frac{x\cdot\nu(y)}{\lambda\, y\cdot\nu(y)}\right) y\cdot\nu(y)\, dH_{n-1}(y) \le n|K|\right\},$$

where $\nu(y)$ is the outer unit normal of ∂K at $y \in \partial K$, where $x \cdot \nu(y)$ denotes the inner product of x and $\nu(y)$, and H_{n-1} is $(n-1)$-dimensional Hausdorff measure. Recall that $\nu(y)$ exists for H_{n-1}-almost all $y \in \partial K$. For the polar (see Section 1 for definitions) of $\Pi_\phi K$ we will write $\Pi^*_\phi K$

When $\phi_1(t) = |t|$, it turns out that for $u \in S^{n-1}$,

$$h_{\Pi_{\phi_1} K}(u) = \frac{1}{|K|}|K_u|,$$

where $|K_u|$ denotes the $(n-1)$-dimensional volume of K_u, the image of the orthogonal projection of K onto the subspace $u^\perp$. Thus

$$\Pi_{\phi_1} K = \frac{c_n}{|K|}\,\Pi K,$$

where ΠK is the classical projection body of K introduced by Minkowski. When $\phi_p(t) = |t|^p$, with $p \ge 1$,

$$\Pi_{\phi_p} K = \frac{c_{n,p}}{|K|^{\frac{1}{p}}}\,\Pi_p K,$$

where $\Pi_p K$ is the L_p projection body of K, defined as the convex body whose support function is given by

$$h_{\Pi_p K}(x) = \left\{\int_{\partial K} |x\cdot\nu(y)|^p |y\cdot\nu(y)|^{1-p}\, dH_{n-1}(y)\right\}^{1/p}.$$

We will prove the following volume ratio inequality.

Theorem. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the volume ratio*

$$\frac{|\Pi^*_\phi K|}{|K|}$$

is maximized when K is an ellipsoid centered at the origin. If ϕ is strictly convex, then ellipsoids centered at the origin are the only maximizers.

When $\phi(t) = |t|$, the theorem is the volume-normalized classical Petty projection inequality. When $\phi(t) = |t|^p$, and $p > 1$, the inequality of the theorem is the L_p Petty projection inequality (established in [33], with an alternate proof given by Campi and Gronchi in [2]). Haberl and Schuster's recent extension [17] of the L_p Petty projection inequality is the case $\phi(t) = (|t| + \alpha t)^p$, for $-1 \le \alpha \le 1$ of the Theorem.

In Section 1, we establish notation and list for quick subsequent reference some basic facts regarding convex functions and convex bodies. In Section 2 some of the basic properties of Orlicz projection bodies are established. Section 3 contains the proof of the Orlicz Petty projection body. In Section 4 some questions are posed.

1. Basics regarding convex bodies

The setting will be Euclidean n-space $\mathbb{R}^n$. We write $e_1, \dots, e_n$ for the standard orthonormal basis of $\mathbb{R}^n$ and when we write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ we always assume that e_n is associated with the last coordinate.

We will attempt to use x, y for vectors in $\mathbb{R}^n$ and x', y' for vectors in $\mathbb{R}^{n-1}$, and $u, v \in S^{n-1}$ for unit vectors. We will use a, b, s, t, α for numbers in $\mathbb{R}$ and c, λ for strictly positive reals. If Q is a Borel subset of $\mathbb{R}^n$ and Q is contained in an i-dimensional affine subspace of $\mathbb{R}^n$ but in no affine subspace of lower dimension, then $|Q|$ will denote the i-dimensional Lebesgue measure of Q. If $x \in \mathbb{R}^n$ then by abuse of notation we will write $|x| = \sqrt{x \cdot x}$.

For $A \in \mathrm{GL}(n)$ write A^t for the transpose of A and A^{-t} for the inverse of the transpose (contragradient) of A. Write $|A|$ for the absolute value of the determinant of A.

We shall write c_n for a constant depending only on n and $c_{n,p}$ for a constant depending only on n and p. For $a \in \mathbb{R}$ define

$$(a)_+ = \max\{a, 0\} \quad \text{and} \quad (a)_- = \min\{a, 0\}.$$

Let $\mathcal{C}$ be the class of convex functions $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$ and such that ϕ is either strictly decreasing on $(-\infty, 0]$ or ϕ is strictly increasing on $[0, \infty)$. We say that the sequence $\phi_i \in \mathcal{C}$ is such that $\phi_i \to \phi_o \in \mathcal{C}$ provided

$$|\phi_i - \phi_o|_I = \max_{t \in I} |\phi_i(t) - \phi_o(t)| \to 0,$$

for every compact interval $I \subset \mathbb{R}$. The sub-class of $\mathcal{C}$ consisting of those $\phi \in \mathcal{C}$ that are strictly convex will be denoted by $\mathcal{C}_s$.

We shall make use of the fact that for $\phi \in \mathcal{C}$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$ with $a_3, a_4 > 0$,

$$(a_3 + a_4)\, \phi\Big(\frac{a_1 + a_2}{a_3 + a_4}\Big) \le a_3\, \phi\Big(\frac{a_1}{a_3}\Big) + a_4\, \phi\Big(\frac{a_2}{a_4}\Big). \tag{1.1}$$

This is a trivial immediate consequence of the convexity of ϕ. If $\phi \in \mathcal{C}_s$ then observe that there is equality in (1.1) if and only if $a_1/a_3 = a_2/a_4$.

Define c_ϕ by

$$c_\phi = \min\{c > 0 : \max\{\phi(c), \phi(-c)\} \le 1\}. \tag{1.2}$$

We write $\mathcal{K}^n$ for the set of convex bodies (compact convex subsets) of $\mathbb{R}^n$. We write $\mathcal{K}^n_o$ for the set of convex bodies that contain the origin in their interiors.

For $K \in \mathcal{K}^n$, let $h(K; \cdot) = h_K : \mathbb{R}^n \to \mathbb{R}$ denote the *support function* of K; i.e., $h(K; x) = \max\{x \cdot y : y \in K\}$. Thus, if $y \in \partial K$, then

$$h_K(\nu_K(y)) = \nu_K(y) \cdot y,$$

where $\nu_K(y)$ denotes the outer unit normal to ∂K at y. We shall make use of the obvious fact that for $c > 0$, for the support function of the convex body $cK = \{cx : x \in K\}$ we have

$$h_{cK} = c h_K. \tag{1.3}$$

Observe that from the definition of the support function it follows immediately that for $A \in \mathrm{GL}(n)$ the support function of the image $AK = \{Ay : y \in K\}$ is given by

$$h_{AK}(x) = h_K(A^t x). \tag{1.4}$$

If $K_i \in \mathcal{K}^n$, we say that $K_i \to K_o \in \mathcal{K}^n$ provided

$$|h_{K_i} - h_{K_o}|_\infty := \max_{u\in S^{n-1}} |h_{K_i}(u) - h_{K_o}(u)| \longrightarrow 0.$$

If $K \in \mathcal{K}_o^n$, then the *polar body* K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It is easy to see that for $c > 0$,

$$(cK)^* = \frac{1}{c} K^*, \tag{1.5}$$

and more generally that for $A \in \mathrm{GL}(n)$

$$(AK)^* = A^{-t}K^*.$$

It is easy to verify that

$$K^{**} = K.$$

We require the easily established continuity of the polar operator $^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$.

Let $\rho(K;\cdot) = \rho_K : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$ denote *radial function* of $K \in \mathcal{K}_o^n$; i.e. $\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}$. It is easily verified that

$$h_{K^*} = 1/\rho_K \qquad \text{and} \qquad \rho_{K^*} = 1/h_K. \tag{1.6}$$

Observe that from the definition of ρ_{K^*} and (1.6) it follows immediately that for $x \in \mathbb{R}^n$

$$h(K,x) = 1 \quad \text{if and only if} \quad x \in \partial K^*. \tag{1.7}$$

The classical Aleksandrov-Fenchel-Jessen *surface area measure*, S_K, of the convex body K can be defined as the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u)\, dS_K(u) = \int_{\partial K} f(\nu_K(y))\, dH_{n-1}(y), \tag{1.8}$$

for each continuous $f : S^{n-1} \to \mathbb{R}$. We shall require the trivial observation that for the surface area measure of cK we have

$$S_{cK} = c^{n-1} S_K, \tag{1.9}$$

and the fact that the measure S_K cannot be concentrated on a hemisphere of S^{n-1}. Slightly less trivial, but much needed is the fact that S_K is weakly continuous in K (see e.g. Schneider [50]); i.e., if $K_i \in \mathcal{K}_o^n$, then

$$K_i \to K_o \in \mathcal{K}_o^n \qquad \Longrightarrow \qquad S_{K_i} \to S_{K_o}, \text{weakly}.$$

That

$$\frac{1}{n}\int_{S^{n-1}} h_K(u)\, dS_K(u) = \frac{1}{n}\int_{\partial K} h_K(\nu_K(y))\, dH_{n-1}(y) = \frac{1}{n}\int_{\partial K} y \cdot \nu_K(y)\, dH_{n-1}(y)$$

is equal to $|K|$ can be easily seen by considering a polytope P in $\mathbb{R}^n$ whose faces have areas (i.e., $(n-1)$-dimensional volumes) $a_1, \ldots, a_m$, corresponding outer unit normals $u_1, \ldots, u_m$, with $V_i = \frac{1}{n} a_i h_P(u_i)$. Here

$$\frac{1}{n}\int_{\partial P} h_P(\nu_P(y))\, dH_{n-1}(y) \;=\; \frac{1}{n}\sum_{i=1}^m a_i h_P(u_i) \;=\; \sum_{i=1}^m V_i \;=\; |P|.$$

For $K \in \mathcal{K}_o^n$, it will be convenient to use volume-normalized *conical measure* V_K defined by

$$|K| dV_K = \frac{1}{n} h_K\, dS_K.$$

Observe that volume-normalized conical measure

$$V_K \text{ is a probability measure on } S^{n-1}. \tag{1.10}$$

For $K \in \mathcal{K}_o^n$, define $R_K, r_K \in (0,\infty)$ by

$$R_K = \max_{u\in S^{n-1}} h_K(u) \qquad \text{and} \qquad r_K = \min_{u\in S^{n-1}} h_K(u). \tag{1.11}$$

It follows from definition (1.8) that, for $u \in S^{n-1}$,

$$\int_{S^{n-1}} (u\cdot v)_+\, dS_K(v) = |K_u| \quad \text{and} \quad \int_{S^{n-1}} (u\cdot v)_-\, dS_K(v) = -|K_u|,$$

where $|K_u|$ denotes the $(n-1)$-dimensional volume of K_u, the image of the orthogonal projection of K onto $u^\perp$. From definition (1.11) we see that the diameter,

$$D_K = \max_{u\in S^{n-1}} [h_K(u) + h_K(-u)],$$

of a body K is at most $2R_K$, and since K is obviously contained in the right cylinder whose base is K_u and whose height is D_K we have the estimates

$$\int_{S^{n-1}} (u\cdot v)_+ \frac{dS_K(v)}{|K|} \ge \frac{1}{2R_K} \quad \text{and} \quad \int_{S^{n-1}} (u\cdot v)_- \frac{dS_K(v)}{|K|} \le -\frac{1}{2R_K}. \tag{1.12}$$

For a convex body $K' \in \mathcal{K}_o^{n-1}$ and a function $g : K' \to \mathbb{R}$ whose gradient exists a.e., define $\langle g\rangle : K' \to \mathbb{R}$ by

$$\langle g\rangle(x') = g(x') - x'\cdot \nabla g(x').$$

We shall often make use of the fact that $\langle\,\cdot\,\rangle$ is a linear operator; i.e., For $g_1, g_2 : K' \to \mathbb{R}$ whose gradient exists a.e., and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\langle \alpha_1 g_1 + \alpha_2 g_2\rangle = \alpha_1\langle g_1\rangle + \alpha_2\langle g_2\rangle. \tag{1.13}$$

For a convex body K and a direction $u \in S^{n-1}$, let $\underline{h}_u(K;\cdot) : K_u \to \mathbb{R}$ and $\overline{h}_u(K;\cdot) : K_u \to \mathbb{R}$ denote the *undergraph* and *overgraph functions* of K with respect to u; i.e.

$$K = \left\{ y' + tu : \ -\underline{h}_u(K;y') \le t \le \overline{h}_u(K;y') \text{ for } y' \in K_u \right\}.$$

Thus, for the *Steiner symmetral*, $\mathrm{S}_u K$, of K in direction u, we see that the image of the orthogonal projections onto $u^\perp$ of both K and $\mathrm{S}_u K$ are identical, and that

$$\underline{h}_u(\mathrm{S}_u K;y') = \frac{1}{2}(\underline{h}_u(K;y') + \overline{h}_u(K;y')) = \overline{h}_u(\mathrm{S}_u K;y'). \tag{1.14}$$

Both K and u will be suppressed when clear from the context, and thus we will often denote the undergraph and overgraph functions of K with respect to u simply by $\underline{h} : K_u \to \mathbb{R}$ and $\overline{h} : K_u \to \mathbb{R}$.

When considering the convex body $K \in \mathcal{K}_o^n$ as $K \subset \mathbb{R}^{n-1}\times\mathbb{R}$, then for $(x',t) \in \mathbb{R}^{n-1}\times\mathbb{R}$ we will usually write $h(K;x',t)$ rather than $h(K;(x',t))$. Note that the Steiner symmetral, $\mathrm{S}_{e_n}K$, of K in the direction e_n can be given by

$$\mathrm{S}_{e_n}K = \{(x', \tfrac{1}{2}t + \tfrac{1}{2}s) \in \mathbb{R}^{n-1}\times\mathbb{R} : (x',t), (x',-s) \in K\}.$$

Finally, we shall make critical use of the following:

Lemma 1.1. *Suppose $K, L \in \mathcal{K}_o^n$ and consider $K, L \subset \mathbb{R}^{n-1} \times \mathbb{R}$. Then*

$$\mathrm{S}_{e_n} K^* \subset L^*,$$

if and only if

$$h(K; y', t) = 1 = h(K; y', -s), \text{ with } t \neq -s \implies h(L; y', \tfrac{1}{2}t + \tfrac{1}{2}s) \leq 1.$$

In addition if $\mathrm{S}_{e_n} K^* = L^*$, then $h(K; y', t) = 1 = h(K; y', -s)$, with $t \neq -s$, always implies that $h(L; y', \frac{1}{2}t + \frac{1}{2}s) = 1$.

Lemma 1.1 is an immediate consequence of the definition of Steiner symmetrization, identities (1.6) and the obvious fact that for each body Q, we have $x \in Q \backslash \partial Q$ if and only if $\rho_Q(x) > 1$.

We say that ∂K, the boundary of K, is *line free* in direction $u \in S^{n-1}$ if $\partial K \cap (x + \mathbb{R}u)$ consists of no more than two points, for each $x \in \partial K$. Note that if ∂K is line free in direction u then $\partial \mathrm{S}_u K$ is line free in direction u. It is known (see [50]) that for each convex body, there is a dense set of directions in which its boundary is line free. In fact, for each convex body K

$$H_{n-1}(\{u \in S^{n-1} : \partial K \text{ is not line free in direction } u\}) = 0.$$

We will make use of the well-known and easily established fact that if ∂K is line free in direction e_n, then for a continuous $g : \partial K \to \mathbb{R}$

(1.15)
$$\int_{\partial K} g(x)\, dH_{n-1}(x) = \int_{K'} g(x', \overline{h}(x'))\sqrt{1 + |\nabla \overline{h}(x')|^2}\, dx' + \int_{K'} g(x', -\underline{h}(x'))\sqrt{1 + |\nabla \underline{h}(x')|^2}\, dx',$$

where $K' = K_{e_n}$.

2. Definition and basic properties of the Orlicz projection bodies

The Orlicz projection body $\Pi_\phi K$ of K is defined as the body whose support function is given by

$$h_{\,\Pi_\phi K}(x) = \min\left\{\lambda > 0 : \int_{\partial K} \phi\left(\frac{x \cdot \nu(y)}{\lambda\, y \cdot \nu(y)}\right) y \cdot \nu(y)\, dH_{n-1}(y) \leq n|K|\right\}, \tag{2.1}$$

where $\nu(y) = \nu_K(y)$ is the outer unit normal of ∂K at $y \in \partial K$, or equivalently, using (1.8),

$$h_{\,\Pi_\phi K}(x) = \min\left\{\lambda > 0 : \int_{S^{n-1}} \phi\left(\frac{x \cdot u}{\lambda\, h_K(u)}\right) h_K(u)\, dS_K(u) \leq n|K|\right\}. \tag{2.2}$$

It will be easier to see the affine nature (Lemma 2.5) of the Orlicz projection body if we use (1.6) to rewrite (2.2) as

$$h_{\,\Pi_\phi K}(x) = \min\left\{\lambda > 0 : \int_{S^{n-1}} \phi(\tfrac{1}{\lambda}(x \cdot u)\rho_{K^*}(u))\, dV_K(u) \leq 1\right\}. \tag{2.3}$$

The polar body of $\Pi_\phi K$ will be denoted by $\Pi_\phi^* K$, rather than $(\Pi_\phi K)^*$.

Since the area measure S_K cannot be concentrated on a closed hemisphere of S^{n-1}, and since we assume that ϕ is strictly increasing on $[0, \infty)$ or strictly decreasing on $(-\infty, 0]$ it follows that the function

$$\lambda \longmapsto \int_{S^{n-1}} \phi(\tfrac{1}{\lambda}(x \cdot u)\rho_{K^*}(u))\, dV_K(u)$$

is strictly decreasing in $(0,\infty)$. Thus, we have:

Lemma 2.1. *Suppose $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$. If $u_o \in S^{n-1}$, then*

$$\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o\, h_K(u)}\right) dV_K(u) = 1$$

if and only if

$$h_{\,\Pi_\phi K}(u_o) \;=\; \lambda_o.$$

We first show that $h_{\,\Pi_\phi K}$ is indeed a support function. It follows immediately from the definition that for all $x \in \mathbb{R}^n$, and for $c > 0$

$$h_{\,\Pi_\phi K}(cx) = c\, h_{\,\Pi_\phi K}(x).$$

We now show that indeed for $x_1, x_2 \in \mathbb{R}^n$,

$$h_{\,\Pi_\phi K}(x_1 + x_2) \;\le\; h_{\,\Pi_\phi K}(x_1) + h_{\,\Pi_\phi K}(x_2).$$

To that end let $h_{\,\Pi_\phi K}(x_i) = \lambda_i$; i.e.,

$$\int_{S^{n-1}} \phi\left(\frac{x_1 \cdot u}{\lambda_1}\rho_{K^*}(u)\right) dV_K(u) = 1 \quad \text{and} \quad \int_{S^{n-1}} \phi\left(\frac{x_2 \cdot u}{\lambda_2}\rho_{K^*}(u)\right) dV_K(u) = 1.$$

The convexity of the function $s \mapsto \phi(s\,\rho_{K^*}(u))$ shows that

$$\phi\left(\frac{x_1 \cdot u + x_2 \cdot u}{\lambda_1 + \lambda_2}\,\rho_{K^*}(u)\right) \le \frac{\lambda_1}{\lambda_1 + \lambda_2}\,\phi\left(\frac{x_1 \cdot u}{\lambda_1}\,\rho_{K^*}(u)\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2}\,\phi\left(\frac{x_2 \cdot u}{\lambda_2}\,\rho_{K^*}(u)\right).$$

Integrating both sides of this inequality with respect to the measure V_K gives us

$$\int_{S^{n-1}} \phi\left(\frac{(x_1 + x_2) \cdot u}{\lambda_1 + \lambda_2}\,\rho_{K^*}(u)\right) dV_K(u) \;\le\; 1, \tag{2.4}$$

from $h_{\,\Pi_\phi K}(x_i) = \lambda_i$. But using definition (2.3), inequality (2.4) gives the desired result that

$$h_{\,\Pi_\phi K}(x_1 + x_2) \;\le\; \lambda_1 + \lambda_2.$$

Thus $h_{\,\Pi_\phi}$ is indeed the support function of a compact convex set. That this set has the origin in its interior (i.e., that $h_{\,\Pi_\phi K}(x) > 0$ whenever $x \ne 0$) follows easily from the fact that either $\lim_{s\to\infty} \phi(s) = \infty$ or $\lim_{s\to-\infty} \phi(s) = \infty$. However we shall require more:

Lemma 2.2. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$, then*

$$\frac{1}{2c_\phi\, R_K} \;\le\; h_{\,\Pi_\phi K}(u) \;\le\; \frac{1}{c_\phi\, r_K},$$

for all $u \in S^{n-1}$

Proof. Suppose $u_o \in S^{n-1}$ and $h_{\,\Pi_\phi K}(u_o) = \lambda_o$; i.e.

$$\frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o\, h_K(u)}\right) \frac{h_K(u)\, dS_K(u)}{|K|} \;=\; 1 \;=\; \int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o\, h_K(u)}\right) dV_K(u).$$

To obtain the lower estimate we proceed as follows. From the definition (1.2), either $\phi(c_\phi) = 1$ or $\phi(-c_\phi) = 1$. Suppose $\phi(-c_\phi) = 1$. Hence from the fact that

ϕ is non-negative, Jensen's inequality, and (1.12) together with the fact that ϕ is monotone decreasing on $(-\infty, 0]$

$$\begin{aligned}\phi(-c_\phi) &= 1\\ &= \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o\, h_K(u)}\right) \frac{h_K(u)\, dS_K(u)}{|K|}\\ &\geq \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{(u_o \cdot u)_-}{\lambda_o\, h_K(u)}\right) \frac{h_K(u)\, dS_K(u)}{|K|}\\ &\geq \phi\left(\frac{1}{n}\int_{S^{n-1}} \frac{(u_o \cdot u)_-}{\lambda_o\, h_K(u)} \frac{h_K(u)\, dS_K(u)}{|K|}\right)\\ &\geq \phi\left(-\frac{1}{2\lambda_o\, R_K}\right).\end{aligned}$$

Since ϕ is monotone decreasing on $(-\infty, 0]$, from this we obtain the lower bound for $h_{\,\Pi_\phi K}$:

$$\frac{1}{2c_\phi\, R_K} \;\leq\; \lambda_o.$$

The case where $\phi(c_\phi) = 1$ is handled the same way and gives the same result.

To obtain the upper estimate, observe that from the definition (1.2), together with the fact that the function $t \mapsto \max\{\phi(t), \phi(-t)\}$ is monotone increasing and definition (1.11), and (1.10) it follows that

$$\begin{aligned}\max\{\phi(c_\phi), \phi(-c_\phi)\} &= 1\\ &= \int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o\, h_K(u)}\right) dV_K(u)\\ &\leq \int_{S^{n-1}} \max\{\phi\left(\frac{|u_o \cdot u|}{\lambda_o\, h_K(u)}\right), \phi\left(\frac{-|u_o \cdot u|}{\lambda_o\, h_K(u)}\right)\}\, dV_K(u)\\ &\leq \int_{S^{n-1}} \max\{\phi\left(\frac{1}{\lambda_o\, h_K(u)}\right), \phi\left(\frac{-1}{\lambda_o\, h_K(u)}\right)\}\, dV_K(u)\\ &\leq \int_{S^{n-1}} \max\{\phi\left(\frac{1}{\lambda_o\, r_K}\right), \phi\left(\frac{-1}{\lambda_o\, r_K}\right)\}\, dV_K(u)\\ &= \max\{\phi\left(\frac{1}{\lambda_o\, r_K}\right), \phi\left(\frac{-1}{\lambda_o\, r_K}\right)\}.\end{aligned}$$

But the even function $t \mapsto \max\{\phi(t), \phi(-t)\}$ is monotone increasing on $[0, \infty)$ so we conclude

$$\lambda_o \;\leq\; \frac{1}{c_\phi\, r_K}.$$

□

For $c > 0$, from (1.3), (1.9) and definition (2.2) we have

$$\Pi_\phi cK \;=\; \frac{1}{c}\,\Pi_\phi K, \tag{2.5}$$

or using (1.5)

$$\Pi^*_\phi cK \;=\; c\,\Pi^*_\phi K. \tag{2.6}$$

The Orlicz projection operator $\Pi_\phi : \mathcal{K}^n_o \to \mathcal{K}^n_o$ is continuous:

Lemma 2.3. *If* $K_i \in \mathcal{K}_o^n$, *then*

$$K_i \to K \in \mathcal{K}_o^n \quad \Longrightarrow \quad \Pi_\phi K_i \to \Pi_\phi K,$$

for each $\phi \in \mathcal{C}$.

Proof. Suppose $u_o \in S^{n-1}$. We will show that

$$h_{\,\Pi_\phi K_i}(u_o) \to h_{\,\Pi_\phi K}(u_o).$$

Let

$$h_{\,\Pi_\phi K_i}(u_o) = \lambda_i,$$

and note that Lemma 2.2 gives

$$\frac{1}{2c_\phi\, R_{K_i}} \;\le\; \lambda_i \;\le\; \frac{1}{c_\phi\, r_{K_i}}.$$

Since $K_i \to K \in \mathcal{K}_o^n$, we have $r_{K_i} \to r_K > 0$ and $R_{K_i} \to R_K < \infty$, and thus there exist a, b such that $0 < a \le \lambda_i \le b < \infty$, for all i. Hence the λ_i have a convergent subsequence, which we also denote by λ_i, such that

$$\lambda_i \to \lambda_o,$$

with $a \le \lambda_o \le b$. Let $\bar{K}_i = \lambda_i K_i$. Since $\lambda_i \to \lambda_o$ and $K_i \to K$, we have

$$\bar{K}_i \to \lambda_o K.$$

Now (2.5), and the fact that $h_{\,\Pi_\phi K_i}(u_o) = \lambda_i$, shows that $h_{\Pi_\phi \bar{K}_i}(u_o) = 1$; i.e.

$$\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{h_{\bar{K}_i}(u)}\right) dV_{\bar{K}_i}(u) = 1,$$

for all i. But $\bar{K}_i \to \lambda_o K$ implies that the functions $h_{\bar{K}_i} \to h_{\lambda_o K}$, uniformly, and the measures $S_{\bar{K}_i} \to S_{\lambda_o K}$, weakly. This in turn implies that the measures $V_{\bar{K}_i} \to V_{\lambda_o K}$, weakly, and hence using the continuity of ϕ we have

$$\int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{h_{\lambda_o K}(u)}\right) dV_{\lambda_o K}(u) = 1,$$

which by Lemma 2.1 gives

$$h_{\Pi_\phi \lambda_o K}(u_o) = 1.$$

Using (2.5) and (1.3), we get from this

$$h_{\Pi_\phi K}(u_o) = \lambda_o = \lim_{i\to\infty} \lambda_i = \lim_{i\to\infty} h_{\,\Pi_\phi K_i}(u_o).$$

What we have in fact shown is that each subsequence of $h_{\,\Pi_\phi K_i}(u_o)$ has a subsequence that converges to $h_{\Pi_\phi K}(u_o)$. And this shows that

$$h_{\,\Pi_\phi K_i}(u_o) \to h_{\,\Pi_\phi K}(u_o),$$

as desired.

Since the support functions $h_{\,\Pi_\phi K_i} \to h_{\,\Pi_\phi K}$ pointwise (on S^{n-1}) they converge uniformly (see e.g., Schneider [50]) completing the proof. □

The continuity of the Orlicz projection operator $\Pi_\phi : \mathcal{K}_o^n \to \mathcal{K}_o^n$ now yields the continuity of the polar Orlicz projection operator $\Pi_\phi^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$.

It turns out that the Orlicz projection body $\Pi_\phi K$ is continuous in ϕ as well as in K:

Lemma 2.4. *If $\phi_i \in \mathcal{C}$, then*

$$\phi_i \to \phi \in \mathcal{C} \qquad \Longrightarrow \qquad \Pi_{\phi_i} K \to \Pi_\phi K,$$

for each $K \in \mathcal{K}_o^n$.

Proof. Suppose $K \in \mathcal{K}_o^n$ and $u_o \in S^{n-1}$. We will show that

$$h_{\,\Pi_{\phi_i} K}(u_o) \to h_{\,\Pi_\phi K}(u_o).$$

Let

$$h_{\,\Pi_{\phi_i} K}(u_o) = \lambda_i,$$

and note that Lemma 2.2 gives

$$\frac{1}{2c_{\phi_i}\, R_K} \;\le\; \lambda_i \;\le\; \frac{1}{c_{\phi_i}\, r_K}.$$

Since $\phi_i \to \phi \in \mathcal{C}$, we have $c_{\phi_i} \to c_\phi \in (0,\infty)$ and thus there exist a, b such that $0 < a \le \lambda_i \le b < \infty$, for all i. Hence the λ_i have a convergent subsequence, which we also denote by λ_i, such that

$$\lambda_i \to \lambda_o,$$

with $a \le \lambda_o \le b$. Since $h_{\,\Pi_{\phi_i} K}(u_o) = \lambda_i$, Lemma 2.1 gives

$$1 = \int_{S^{n-1}} \phi_i\left(\frac{u_o \cdot u}{\lambda_i h_K(u)}\right) dV_K(u).$$

Hence

$$1 = \int_{S^{n-1}} \phi\left(\frac{u_o \cdot u}{\lambda_o h_K(u)}\right) dV_K(u),$$

which by Lemma 2.1 gives

$$h_{\Pi_\phi K}(u_o) = \lambda_o,$$

and thus

$$h_{\Pi_\phi K}(u_o) = \lambda_o = \lim_{i\to\infty} \lambda_i = \lim_{i\to\infty} h_{\,\Pi_{\phi_i} K}(u_o).$$

What we have in fact shown is that each subsequence of $h(\,\Pi_{\phi_i} K; u_o)$ has a subsequence that converges to $h(\Pi_\phi K; u_o)$. And this shows that $h(\Pi_{\phi_i} K; u_o) \to h(\Pi_\phi K; u_o)$ as desired.

Since the support functions $h_{\,\Pi_{\phi_i} K} \to h_{\,\Pi_\phi K}$ pointwise (on S^{n-1}) they converge uniformly and hence

$$\Pi_{\phi_i} K \to \Pi_\phi K.$$

□

The affine nature of the Orlicz projection body is revealed in:

Lemma 2.5. *If $K \in \mathcal{K}_o^n$ and $A \in \mathrm{SL}(n)$, then*

$$\Pi_\phi A K = A^{-t} \Pi_\phi K$$

Suppose P is a polytope whose $(n-1)$-dimensional faces are $F_1, \ldots, F_m$. Let $u_1, \ldots, u_m$ be the outer unit normals to the faces, and let $h_1, \ldots, h_m$ denote *support numbers* of the faces; i.e., $h(P; u_i) = h_i$. Let $V_1, \ldots, V_m$ denote the volumes of the *facial cones* (i.e., the volumes of the cones that the faces of P form with the origin as the vertex); i.e., $V_i = \frac{1}{n} h_i |F_i|$. Finally, let V denote the volume of the polytope P.

For $A \in \mathrm{SL}(n)$, let $P^\star = AP = \{Ax : x \in P\}$. Let $F_1^\star, \ldots, F_m^\star$ denote the faces of $P^\star$, let $u_1^\star, \ldots, u_m^\star$ be the outer unit normals of the faces of $P^\star$ and let $h_1^\star, \ldots, h_m^\star$

denote the corresponding support numbers of $P^\star$. Since $A \in \mathrm{SL}(n)$, obviously the volumes $V_1^\star, \dots, V_m^\star$ of the facial cones of $P^\star$ are such that $V_i^\star = V_i$.

The face F_i parallel to the subspace $u_i^\perp$ is transformed by A into the face $F_i^\star = AF_i$ parallel to $(A^{-t}u_i)^\perp$ and thus

$$u_i^\star \;=\; A^{-t}u_i/|A^{-t}u_i|. \tag{2.7}$$

Now $h_i^\star = h(P^\star, u_i^\star) = h(AP, u_i^\star) = h(P, A^t u_i^\star)$, by (1.4). Thus, from (2.7) we have

$$h_i^\star = h(P, A^t u_i^\star) = h(P, u_i/|A^{-t}u_i|) = h(P, u_i)/|A^{-t}u_i| = h_i/|A^{-t}u_i|. \tag{2.8}$$

Now from definition (2.3), the fact that $V^\star = V$ and $V_i^\star = V_i$ together with (2.7) and (2.8), definition (2.3) again, and finally (1.4), we have

$$\begin{aligned}
h_{\Pi_\phi AP}(x) &= h_{\Pi_\phi P^\star}(x) \\
&= \min\left\{\lambda > 0 : \sum_{i=1}^m \phi\Big(\frac{x \cdot u_i^\star}{\lambda h_i^\star}\Big)\,\frac{V_i^\star}{V^\star} \;\le\; 1\right\} \\
&= \min\left\{\lambda > 0 : \sum_{i=1}^m \phi\Big(\frac{x \cdot A^{-t}u_i}{\lambda h_i}\Big)\,\frac{V_i}{V} \;\le\; 1\right\} \\
&= \min\left\{\lambda > 0 : \sum_{i=1}^m \phi\Big(\frac{A^{-1}x \cdot u_i}{\lambda h_i}\Big)\,\frac{V_i}{V} \;\le 1\right\} \\
&= h_{\Pi_\phi P}(A^{-1}x) \\
&= h_{A^{-t}\Pi_\phi P}(x),
\end{aligned}$$

showing that $\Pi_\phi AP = A^{-t}\Pi_\phi P$. This along with Lemma 2.3 completes the proof of Lemma 2.5.

3. Proof of the Orlicz Petty projection inequality

We shall require:

Lemma 3.1. *If $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is a convex body that contains the origin in its interior, and if ∂K is line free in direction e_n, then for $(y', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$,*

$$\begin{aligned}
&\int_{\partial K} \phi\left(\frac{(y',t)\cdot \nu_K(x)}{x \cdot \nu_K(x)}\right) x \cdot \nu_K(x)\, dH_{n-1}(x) \\
&= \int_{K'} \phi\left(\frac{t - y' \cdot \nabla \overline{h}(x')}{\langle\, \overline{h}\,\rangle(x')}\right) \langle\, \overline{h}\,\rangle(x')\, dx' + \int_{K'} \phi\left(\frac{-t - y' \cdot \nabla \underline{h}(x')}{\langle\, \underline{h}\,\rangle(x')}\right) \langle\, \underline{h}\,\rangle(x')\, dx',
\end{aligned}$$

were $K' = K_{e_n}$ denotes the image projection of K onto the subspace $e_n^\perp = \mathbb{R}^{n-1}$.

Proof. Note that we abbreviated the overgraph and undergraph functions of K in the direction e_n by $\overline{h} = \overline{h}_{e_n}(K;\cdot) : K' \to \mathbb{R}$ and $\underline{h} = \underline{h}_{e_n}(K;\cdot) : K' \to \mathbb{R}$; i.e.,

$$K = \{x' + se_n : x' \in K',\ -\underline{h}(x') \le s \le \overline{h}(x')\}.$$

For $x' \in K'$, denote the outer unit normal of the upper graph of K at $(x', \overline{h}(x'))$ by $\overline{\nu}(x')$. Thus,

$$\overline{\nu}(x') = \frac{(-\nabla \overline{h}(x'), 1)}{(1 + |\nabla \overline{h}(x')|^2)^{\frac{1}{2}}}.$$

Denote the outer unit normal of the lower graph of K at $(x', -\underline{h}(x'))$ by $\underline{\nu}(x')$, and have

$$\underline{\nu}(x') = \frac{(-\nabla \underline{h}(x'), -1)}{(1 + |\nabla \underline{h}(x')|^2)^{\frac{1}{2}}}.$$

When x is on the upper graph of ∂K, i.e., $x = (x', \overline{h}(x'))$, we have

$$x \cdot \nu_K(x) = (x', \overline{h}(x')) \cdot \overline{\nu}(x') = \frac{\langle \overline{h} \rangle (x')}{(1 + |\nabla \overline{h}(x')|^2)^{\frac{1}{2}}}, \tag{3.1}$$

When x is on the lower graph of ∂K, i.e., $x = (x', -\underline{h}(x'))$, we have

$$x \cdot \nu_K(x) = (x', -\underline{h}(x')) \cdot \underline{\nu}(x') = \frac{\langle \underline{h} \rangle (x')}{(1 + |\nabla \underline{h}(x')|^2)^{\frac{1}{2}}}, \tag{3.2}$$

To complete the proof we now appeal to (1.15). □

The main ingredient of the proof of the Theorem is:

Proposition. *Suppose $\phi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$ and ∂K is line free in direction u, then*

$$\mathrm{S}_u \Pi_\phi^* K \subseteq \Pi_\phi^*(\mathrm{S}_u K). \tag{3.3}$$

If $\phi \in \mathcal{C}_s$ and $\mathrm{S}_u \Pi_\phi^* K = \Pi_\phi^*(\mathrm{S}_u K)$, then all of the midpoints of the chords of K parallel to u lie on a hyperplane that passes through the origin.

Proof. Without loss of generality, assume that $u = e_n$. We will be appealing to Lemma 1.1 and thus we begin by supposing that

$$h(\Pi_\phi K; y', t) = 1, \qquad \text{and} \qquad h(\Pi_\phi K; y', -s) = 1,$$

with $t \neq -s$, or equivalently, by (1.7), that

$$(y', t) \in \partial \Pi_\phi^* K, \qquad \text{and} \qquad (y', -s) \in \partial \Pi_\phi^* K.$$

By Lemma 2.1 this means that

$$\frac{1}{n|K|} \int_{\partial K} \phi\left(\frac{(y', t) \cdot \nu_K(x)}{x \cdot \nu_K(x)}\right) x \cdot \nu_K(x)\, dH_{n-1}(x) = 1 \tag{3.4a}$$

and

$$\frac{1}{n|K|} \int_{\partial K} \phi\left(\frac{(y', -s) \cdot \nu_K(x)}{x \cdot \nu_K(x)}\right) x \cdot \nu_K(x)\, dH_{n-1}(x) = 1. \tag{3.4b}$$

By Lemma 1.1, the desired inclusion (3.3) will have been established if we can show that

$$h(\Pi_\phi \mathrm{S}_u K; y', \tfrac{1}{2}t + \tfrac{1}{2}s) \leq 1. \tag{3.5}$$

By Lemma 3.1 and (1.14), (1.13), (1.1), and Lemma 3.1 once again, we have

(3.6)

$$
\begin{aligned}
&\int_{\partial(\mathrm{S}_u K)} \phi\left(\frac{(y', \frac12 t + \frac12 s)\cdot \nu_{\mathrm{S}_u K}(x)}{x\cdot \nu_{\mathrm{S}_u K}(x)}\right) x\cdot \nu_{\mathrm{S}_u K}(x)\, dH_{n-1}(x)\\
=&\int_{K'} \phi\left(\frac{\frac12 t + \frac12 s - y'\cdot\nabla(\frac12 \underline{h} + \frac12 \overline{h})(x')}{\langle \frac12 \underline{h} + \frac12 \overline{h}\,\rangle(x')}\right)\langle \tfrac12 \underline{h} + \tfrac12 \overline{h}\,\rangle(x')\, dx'\\
&\quad + \int_{K'} \phi\left(\frac{-\frac12 t - \frac12 s - y'\cdot\nabla(\frac12 \underline{h} + \frac12 \overline{h})(x')}{\langle \frac12 \underline{h} + \frac12 \overline{h}\,\rangle(x')}\right)\langle \tfrac12 \underline{h} + \tfrac12 \overline{h}\,\rangle(x')\, dx'\\
\le&\frac12 \int_{K'} \phi\left(\frac{t - y'\cdot\nabla\overline{h}(x')}{\langle\, \overline{h}\,\rangle(x')}\right)\langle\, \overline{h}\,\rangle(x')\, dx' + \frac12 \int_{K'} \phi\left(\frac{s - y'\cdot\nabla\underline{h}(x')}{\langle\, \underline{h}\,\rangle(x')}\right)\langle\, \underline{h}\,\rangle(x')\, dx'\\
&\quad + \frac12 \int_{K'} \phi\left(\frac{-t - y'\cdot\nabla\underline{h}(x')}{\langle\, \underline{h}\,\rangle(x')}\right)\langle\, \underline{h}\,\rangle(x')\, dx' + \frac12 \int_{K'} \phi\left(\frac{-s - y'\cdot\nabla\overline{h}(x')}{\langle\, \overline{h}\,\rangle(x')}\right)\langle\, \overline{h}\,\rangle(x')\, dx'\\
=&\frac12 \int_{\partial K} \phi\left(\frac{(y', t)\cdot \nu_K(x)}{x\cdot \nu_K(x)}\right) x\cdot \nu_K(x)\, dH_{n-1}(x)\\
&\quad + \frac12 \int_{\partial K} \phi\left(\frac{(y', -s)\cdot \nu_K(x)}{x\cdot \nu_K(x)}\right) x\cdot \nu_K(x)\, dH_{n-1}(x).
\end{aligned}
$$

As an aside, observe that if ϕ is strictly convex, then (1.1) tells us that equality in (3.6) would imply that

$$
\frac{t - y'\cdot\nabla\overline{h}(x')}{\langle\, \overline{h}\,\rangle(x')} = \frac{s - y'\cdot\nabla\underline{h}(x')}{\langle\, \underline{h}\,\rangle(x')} \qquad \text{and} \qquad \frac{-s - y'\cdot\nabla\overline{h}(x')}{\langle\, \overline{h}\,\rangle(x')} = \frac{-t - y'\cdot\nabla\underline{h}(x')}{\langle\, \underline{h}\,\rangle(x')},
$$

for all $x' \in K'$.

Since $|\mathrm{S}_u K| = |K|$, it follows from (3.4) and (3.6), that

$$
\frac{1}{n|\mathrm{S}_u K|}\int_{\partial(\mathrm{S}_u K)} \phi\left(\frac{(y', \frac12 t + \frac12 s)\cdot \nu_{\mathrm{S}_u K}(x)}{x\cdot \nu_{\mathrm{S}_u K}(x)}\right) x\cdot \nu_{\mathrm{S}_u K}(x)\, dH_{n-1}(x) \ \le 1.
$$

This and a glance at definition (2.1), gives (3.5), and thus (3.3) is proved.

Suppose that ϕ is strictly convex and

$$
\mathrm{S}_u \Pi^*_\phi K = \Pi^*_\phi(\mathrm{S}_u K). \tag{3.7}
$$

For each $y' \in K'$, that is sufficiently close to the origin, there exist real $t_{y'}$ and $s_{y'}$, with $t_{y'} \neq -s_{y'}$, such that

$$
h(\Pi_\phi K; y', t_{y'}) = 1, \qquad \text{and} \qquad h(\Pi_\phi K; y', -s_{y'}) = 1; \tag{3.8}
$$

or equivalently by (1.7)

$$
(y', t_{y'}) \in \partial\Pi^*_\phi K, \qquad \text{and} \qquad (y', -s_{y'}) \in \partial\Pi^*_\phi K. \tag{3.9}
$$

By Lemma 1.1, (3.7) and (3.8) forces.

$$
h(\Pi_\phi \mathrm{S}_u K; y', \tfrac12 t_{y'} + \tfrac12 s_{y'}) = 1 \tag{3.10}
$$

Now (3.10) forces equality in (3.6). The strict convexity of ϕ and the equality conditions of (1.1) now show that

$$\frac{t_{y'} - y' \cdot \nabla\overline{h}(x')}{\langle\, \overline{h}\,\rangle(x')} = \frac{s_{y'} - y' \cdot \nabla\underline{h}(x')}{\langle\, \underline{h}\,\rangle(x')}, \tag{3.11a}$$

$$\frac{-s_{y'} - y' \cdot \nabla\overline{h}(x')}{\langle\, \overline{h}\,\rangle(x')} = \frac{-t_{y'} - y' \cdot \nabla\underline{h}(x')}{\langle\, \underline{h}\,\rangle(x')}, \tag{3.11b}$$

for all $x' \in K'$. Let $y' = 0$ and note from (3.9) that $s_0 \neq 0$ and $t_0 \neq 0$. Observe that the denominators in (3.11) are strictly positive for all $x' \in K'$, and by solving (3.11) we see that

$$\langle\, \overline{h}\,\rangle(x') = \langle\, \underline{h}\,\rangle(x'),$$

for all $x' \in K'$; i.e.,

$$(\overline{h} - \underline{h})(x') - x' \cdot \nabla(\overline{h} - \underline{h})(x') = 0 \tag{3.12}$$

for all $x' \in K'$. But the fact that $\langle\, \overline{h}\,\rangle(x') = \langle\, \underline{h}\,\rangle(x')$, for all $x' \in K'$, together with (3.11) shows that

$$(t_{y'} - s_{y'}) - y' \cdot \nabla(\overline{h} - \underline{h})(x') = 0, \tag{3.13}$$

for all $x' \in K'$. However (3.13) says that

$$\{\nabla(\overline{h} - \underline{h})(x') : x' \in K'\}$$

is a set of points that must lie in a plane of $\mathbb{R}^{n-1}$ with normal vector y'. But y' can be chosen in any direction in $\mathbb{R}^{n-1}$, thus there exists an $x'_o \in \mathbb{R}^{n-1}$

$$\{\nabla(\overline{h} - \underline{h})(x') : x' \in K'\} = \{x'_o\}.$$

Substituting this into (3.12) shows that

$$(\overline{h} - \underline{h})(x') = x' \cdot x'_o$$

for all $x' \in K'$. And this shows that the midpoints of the chords of K parallel to e_n,

$$\{(x', \tfrac{1}{2}\overline{h}(x') - \tfrac{1}{2}\underline{h}(x')) : x' \in K'\},$$

lie in the subspace $\{(x', \frac{1}{2}x'_o \cdot x') : x' \in \mathbb{R}^{n-1}\}$ of $\mathbb{R}^n$. □

If $u_i \in S^{n-1}$ are such that $u_i \to u$, then $\mathrm{S}_{u_i}L \to \mathrm{S}_u L$, for each $L \in \mathcal{K}^n_o$. This observation, the continuity of $\Pi^*_\phi : \mathcal{K}^n_o \to \mathcal{K}^n_o$, and the fact that every convex body has a boundary that's line free in almost all directions, allows us to conclude from the Proposition:

Corollary 3.1. *If $\phi \in \mathcal{C}$, and $K \in \mathcal{K}^n_o$, then*

$$\mathrm{S}_u \Pi^*_\phi K \subseteq \Pi^*_\phi(\mathrm{S}_u K),$$

for all $u \in S^{n-1}$.

To establish the equality conditions of our theorem will make use of the following classical characterization of ellipsoids centered at the origin: A convex body $K \in \mathcal{K}^n_o$ is an ellipsoid centered at the origin if and only if there exists a dense set of directions $D \subset S^{n-1}$ such that for each $u \in D$, the midpoints of the chords of K parallel to u lie in a subspace of $\mathbb{R}^n$.

Theorem. *Suppose $\phi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$, then the volume ratio*

$$\frac{|\Pi_\phi^* K|}{|K|}$$

is maximized when K is an ellipsoid centered at the origin. If $\phi \in \mathcal{C}_s$, then ellipsoids centered at the origin are the only maximizers.

Proof. Suppose $\phi \in \mathcal{C}_s$ and K is not an ellipsoid centered at the origin. Choose a direction u in which ∂K is line-free and for which the chords of K (in direction u) have midpoints which do not lie in a subspace of $\mathbb{R}^n$. Let $K_1 = \mathrm{S}_u K$. From the Proposition, and the fact that Steiner symmetrization leaves volume unchanged, it follows that

$$|\Pi_\phi^* K| < |\Pi_\phi^* K_1| \qquad \text{with} \qquad |K| = |K_1|.$$

Choose a random line-free direction u_i for ∂K_i, let $K_{i+1} = \mathrm{S}_{u_i} K_i$, and we obtain a sequence $K_i \to cB$, with $c > 0$, which by Proposition, and the continuity of $\Pi_\phi^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$ satisfy

$$|\Pi_\phi^* K| < |\Pi_\phi^* K_1| \leq \cdots \leq |\Pi_\phi^* K_i| \ \to \ |\Pi_\phi^* cB|,$$

with $|K| = |K_i|$ and thus $|K| = |cB|$. From (2.6) we know that $\Pi_\phi^* cB = c\,\Pi_\phi^* B$ and from this we have

$$\frac{|\Pi_\phi^* K|}{|K|} < \frac{|\Pi_\phi^* B|}{|B|}.$$

In the above argument we have made use of the fact that a sequence of Steiner symmetrizations in directions chosen at random will converge to a ball.

For $\phi \in \mathcal{C}$ that's not necessarily strictly convex, use the same argument — but now with a first step that results in an inequality that's not necessarily strict. □

4. Open Problem

The equality conditions in the Theorem were only established under the assumption that ϕ is strictly convex. Was this restriction necessitated by our methods?

Conjecture 4.1. *Suppose $\phi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$, then the volume ratio*

$$\frac{|\Pi_\phi^* K|}{|K|}$$

is maximized only when K is an ellipsoid.

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