L_p JOHN ELLIPSOIDS

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Abstract. It is shown that the classical John ellipsoid, the Petty ellipsoid and a recently discovered 'dual' of the Legendre ellipsoid are all special cases ($p = \infty$, $p = 1$, and $p = 2$) of a family of L_p ellipsoids which can be associated with a fixed convex body. This insight allows for a unified view of, alternate approaches to, and extensions of some basic results in convex geometry.

Introduction

Two old questions in convex geometry ask: (1) What is the largest (in volume) ellipsoid that can be squeezed inside a fixed convex body? (2) When $\mathrm{SL}(n)$ transformations act on a fixed convex body in $\mathbb{R}^n$, which transformation yields the image with the smallest surface area? One of the aims of this article is to demonstrate that these apparently unrelated questions are special cases of the same problem – that of minimizing *total L_p-curvature*. Problem (1) turns out to be the L_∞ case while Problem (2) is the L_1 case.

An often used fact in both convex and Banach space geometry is that associated with each convex body K is a unique ellipsoid $\mathrm{J}K$ of maximal volume contained in K. The ellipsoid is called the *John ellipsoid* (or *Löwner-John ellipsoid*) of K and the center of this ellipsoid is called the *John point* of the body K. The John ellipsoid is extremely useful, see, for example, [2] and [40] for applications.

Two important results concerning the John ellipsoid are John's inclusion and Ball's volume-ratio inequality. *John's inclusion* states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$K \subseteq \sqrt{n}\, \mathrm{J}K. \tag{0.1}$$

Among a slew of applications, John's inclusion gives the best upper bound, $\sqrt{n}$, for the Banach-Mazur distance of an n-dimensional normed space to n-dimensional Euclidean

1991 *Mathematics Subject Classification.* MR 52A40.

Research supported, in part, by NSF Grant DMS–9803261 and DMS–0104363

space. *Ball's volume-ratio inequality* is the following: If K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$|K|/|\,\mathrm{J}K| \;\leq\; 2^n/\omega_n, \tag{0.2}$$

with equality if and only if K is a parallelotope. Here $|\cdot|$ denotes n-dimensional volume and $\omega_n = \pi^{n/2}\Gamma(1+n/2)$ denotes the volume of the unit ball, B, in $\mathbb{R}^n$. The fact that there is equality in (0.2) only for parallelotopes was only recently established by Barthe [4].

The authors recently introduced in [31] a new ellipsoid $\Gamma_{-2}K$ associated with each convex body K that contains the origin in its interior. The volume of the ellipsoid $\Gamma_{-2}K$ is dominated by the volume of K. It was proved in [31] that for the new ellipsoid there is an inclusion identical to John's inclusion: If K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$K \;\subseteq\; \sqrt{n}\,\Gamma_{-2}K. \tag{0.3}$$

It was also shown in [31] that for the new ellipsoid Ball's volume-ratio inequality (0.2) continues to hold: If K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$|K|/|\Gamma_{-2}K| \leq 2^n/\omega_n, \tag{0.4}$$

with equality if and only if K is a parallelotope.

Unlike the John ellipsoid, for the new ellipsoid there is an analytic formulation. If $x \in \mathbb{R}^n$,

$$\|x\|^2_{\Gamma_{-2}K} = \frac{1}{|K|}\int_{S^{n-1}} |x\cdot u|^2 dS_2(K,u) \tag{0.5}$$

where $S_2(K,\cdot)$ is the quadratic surface are measure of K (as defined in Section 1).

In this paper, we show that associated with each convex body K, that contains the origin in its interior, is a family of ellipsoids E_pK, the L_p *John ellipsoids* of K. For origin-symmetric K, the ellipsoid $\mathrm{E}_\infty K$ turns out to be the classical John ellipsoid. In the L_2 case, E_2K is the new ellipsoid $\Gamma_{-2}K$. The ellipsoid E_1K is the *Petty ellipsoid* of K. The volume-normalized Petty ellipsoid is obtained by minimizing the surface area of K under $\mathrm{SL}(n)$ transformations of K. See Petty [39] and also Giannopoulos and Papadimitrakis [13] and [29]. (Petty [39] contains references to work done on this minimization problem in the first half of the Twentieth Century.)

In Section 4, we shall present an L_p version of John's inclusion. This yields an alternate approach to obtaining Lewis' upper bound for the Banach-Mazur distance between n-dimensional subspaces of an L_p space and Euclidean n-space.

In Section 5, we show that Ball's volume-ratio inequality holds not only for the John ellipsoid, but for all the L_p John ellipsoids (for all $p \in (0,\infty]$).

Lewis [22] showed that associated with each n-dimensional subspace of L_p is an important ellipsoid. These *Lewis ellipsoids* have become basic tools in Banach space geometry. The view taken in this paper is very different from that of Lewis: We associate a family of ellipsoids with one n-dimensional Banach space. A simple way to observe the different results emerging from these different approaches is to consider the L_2 case. Here the Lewis ellipsoid is trivial. By contrast, in our approach it is precisely the L_2 case that yields one of the most important of the L_p John ellipsoid – the recently discovered ellipsoid that is dual to the Legendre ellipsoid of classical mechanics. See [31], [32], and Ludwig [25].

Much effort has been expended in obtaining the results concerning L_p John ellipsoids for all $p > 0$. Many of the arguments could have been greatly simplified had these results only been desired for $p \geq 1$.

There is a bit of overlap between our work and that of Bastero and Romance [5]. However, [5] is mainly concerned with associated ellipsoids in the dual Brunn-Minkowski theory (see e.g. [10] for a discussion of the relationship between the classical and dual Brunn-Minkowski theories). Ellipsoids associated with convex bodies are also studied in interesting recent work of Klartag [20].

Acknowledgement The authors are grateful to the referee for many suggested improvements and for the thoughtful and careful reading given to the original draft of this paper.

1. Minimizing total L_p-curvature

For quick reference we recall some basic results from the Brunn-Minkowski theory. Good references are Gardner [G], Leichtweiß [21], Schneider [44], and Thompson [49].

A *convex body* in Euclidean n-dimensional space, $\mathbb{R}^n$, is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. For a convex body Q let $h_Q : \mathbb{R}^n \to \mathbb{R}$ denote its *support function*; i.e., for $x \in \mathbb{R}^n$, we have $h_Q(x) = \max\{x \cdot y \colon y \in K\}$, where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. If $\phi \in \mathrm{GL}(n)$, then for the support function of the image $\phi K = \{\phi x : x \in K\}$, we obviously have

$$h_{\phi Q}(x) = h_Q(\phi^t x), \tag{1.1}$$

where ϕ^t denotes the transpose of ϕ. If Q contains the origin in its interior, then we will use Q^* to denote the polar of Q; i.e.,

$$Q^* = \{x \in \mathbb{R}^n \colon x \cdot y \leq 1 \text{ for all } y \in Q\}.$$

Obviously, for $\phi \in \mathrm{GL}(n)$,

$$(\phi Q)^* = \phi^{-t} Q^*, \tag{1.2}$$

where ϕ^{-t} denotes the inverse of the transpose of ϕ.

The *radial* function $\rho_Q \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ associated with a set $Q \subset \mathbb{R}^n$ that is compact and star-shaped is defined for $x \neq 0$ by $\rho_Q(x) = \max\{\lambda \geq 0 \colon \lambda x \in Q\}$. If ρ_Q is positive and continuous, Q is called a *star body*. Obviously, for $x \neq 0$ and $\phi \in \mathrm{SL}(n)$,

$$\rho_{\phi Q}(x) = \rho_Q(\phi^{-1} x). \tag{1.3}$$

It is easily seen that if Q is a convex body in $\mathbb{R}^n$, then $\rho_{Q^*} = 1/h_Q$ and if in addition Q is origin symmetric then

$$\rho_Q(u) = \min\{h_Q(v)/|u \cdot v| : v \in S^{n-1}\}, \tag{1.4}$$

for all $u \in S^{n-1}$.

Throughout, the letter E will be used exclusively to denote an origin-symmetric ellipsoid.

Recall that the classical *surface area measure*, $S(K,\cdot)$, of a convex body, K, is a Borel measure on the unit sphere S^{n-1} defined so that

$$\lim_{\varepsilon\to 0^+} \frac{|K+\varepsilon Q|-|K|}{\varepsilon} = \int_{S^{n-1}} h(Q,u)\, dS(K,u), \tag{1.5}$$

hold for each convex body Q. Here $|\ \cdot\ |$ denotes n-dimensional volume and $K+\varepsilon Q = \{x+\varepsilon y \colon x \in K \text{ and } y \in Q\}$ is the standard Minkowski sum of the body K and the dilate $\varepsilon Q = \{\varepsilon x : x \in Q\}$. If the measure $S(K,\ \cdot\ )$ is absolutely continuous with respect to standard Lebesgue measure, S, on S^{n-1}, then it is said that K has a *curvature function*, $f(K,\ \cdot\ )\colon S^{n-1} \to \mathbb{R}$, defined as the Radon-Nikodym derivative $f(K,\ \cdot\ ) = dS(K,\ \cdot\ )/dS$. From definition (1.5) it follows that if K, L are convex bodies and $\phi \in \mathrm{SL}(n)$, then

$$\int_{S^{n-1}} h_{\phi L}(u)\, dS(\phi K, u) = \int_{S^{n-1}} h_L(u)\, dS(K,u). \tag{1.6}$$

If the body K contains the origin in its interior, then for each real p, we can define $dS_p(K,\ \cdot\ )$, the L_p-*surface area measure* of K by:

$$dS_p(K,\ \cdot\ ) = h_K^{1-p} dS(K,\ \cdot\ ). \tag{1.7}$$

If $\lambda > 0$, then for the dilate λK, we know $h_{\lambda K} = \lambda h_K$ and $S(\lambda K,\ \cdot\ ) = \lambda^{n-1} S(K,\ \cdot\ )$. It follows immediately from (1.7) that

$$S_p(\lambda K,\ \cdot\ ) = \lambda^{n-p} S_p(K,\ \cdot\ ). \tag{1.8}$$

If in addition to containing the origin in its interior, the body K has a curvature function, then $f_p(K,\ \cdot\ )\colon S^{n-1} \to \mathbb{R}$, the L_p-*curvature function* of K is defined by:

$$f_p(K,\ \cdot\ ) = h_K^{1-p}\, f(K,\ \cdot\ ).$$

Since its introduction in the mid 90's, L_p curvature (and the functionals it gives rise to) have attracted increased interest (see, e.g., Campi and Gronchi [7,8], Chou and Wang [9], Gardner [11], Guan and Lin [15], Hug and Schneider [18], Klain [19], Ludwig [24,25,26], Meyer and Werner [38], Ryabogin and Zvavitch [43], Schütt and Werner [45, 46], Stancu [47, 48], Umanskiy [50], Werner [51], and also [17], [27,28,29], [33,34,35] and [30].)

In this paper we will be interested in minimizing total L_p-curvature of a body under $\mathrm{SL}(n)$-transformations of the body: Given a smooth convex body K in $\mathbb{R}^n$, that contains the origin in its interior, and a fixed real $p > 0$, find

$$\min_{\phi \in \mathrm{SL}(n)} \int_{S^{n-1}} f_p(\phi K, u)\, dS(u).$$

That this minimum is actually attained is easy to see (and will be shown in Section 2). A $\phi_p \in \mathrm{SL}(n)$ at which this minimum is attained defines an ellipsoid $\bar{\mathrm{E}}_p K$ which ϕ_p maps into the unit ball, B; i.e., $\bar{\mathrm{E}}_p K = \phi_p^{-1} B$. This ellipsoid is unique and will be called the *volume-normalized L_p John ellipsoid* of K. For $p = \infty$, define

$$\bar{\mathrm{E}}_\infty K = \lim_{p \to \infty} \bar{\mathrm{E}}_p K.$$

It will be helpful to introduce some additional notation: For $x \in \mathbb{R}^n$, let $\langle x \rangle = x/|x|$, whenever $x \neq 0$. We shall use $e_1, \dots, e_n$ to denote the canonical basis for $\mathbb{R}^n$.

Definition 1.1. *Given a measure $d\mu(u)$ on S^{n-1}, a real $p > 0$, and a $\phi \in \mathrm{GL}(n)$, define the measure $d\mu^{(p)}(\phi u)$ on S^{n-1} by*

$$\int_{S^{n-1}} f(u)\, d\mu^{(p)}(\phi u) = \int_{S^{n-1}} |\phi^{-1} u|^p f(\langle \phi^{-1} u \rangle)\, d\mu(u),$$

for each $f \in C(S^{n-1})$.

First note that for each body K and each $\phi \in \mathrm{SL}(n)$ for the classical surface area measure we have:

$$dS(\phi K, u) = dS^{(1)}(K, \phi^t u). \tag{1.9}$$

To see this note that for each convex body Q it follows from Definition 1.1, the homogeneity of h_Q, (1.1) and (1.6) that

$$\begin{aligned}
\int_{S^{n-1}} h_Q(u)\, dS^{(1)}(K, \phi^t u) &= \int_{S^{n-1}} |\phi^{-t} u| h_Q(\langle \phi^{-t} u \rangle)\, dS(K, u), \\
&= \int_{S^{n-1}} h_Q(\phi^{-t} u)\, dS(K, u) \\
&= \int_{S^{n-1}} h_{\phi^{-1} Q}(u)\, dS(K, u) \\
&= \int_{S^{n-1}} h_Q(u)\, dS(\phi K, u).
\end{aligned}$$

As is the case above, we shall make use of the fact that if two Borel measures on S^{n-1} are equal when integrated against support functions of convex bodies, then the measures are identical. This follows from the fact that the differences of support functions are dense in $C(S^{n-1})$, with the sup norm.

Proposition 1.2. *If K is a convex body that contains the origin in its interior and real $p > 0$, then for $\phi \in \mathrm{SL}(n)$,*

$$dS_p(\phi K, u) = dS_p^{(p)}(K, \phi^t u). \tag{1.10}$$

Proof. If $f \in C(S^{n-1})$, then from definition (1.7), (1.1), (1.9), Definition 1.1, the homogeneity of h_K, definition (1.7) again, and Definition 1.1 again, we have:

$$\begin{aligned}\int_{S^{n-1}} f(u) dS_p(\phi K, u) &= \int_{S^{n-1}} f(u) h_K^{1-p}(\phi^t u)\, dS(\phi K, u)\\ &= \int_{S^{n-1}} f(u) h_K^{1-p}(\phi^t u)\, dS^{(1)}(K, \phi^t u)\\ &= \int_{S^{n-1}} |\phi^{-t} u|\, f(\langle \phi^{-t} u\rangle) h_K^{1-p}(\phi^t \langle \phi^{-t} u\rangle)\, dS(K, u)\\ &= \int_{S^{n-1}} |\phi^{-t} u|^p f(\langle \phi^{-t} u\rangle)\, dS_p(K, u)\\ &= \int_{S^{n-1}} f(u)\, dS_p^{(p)}(K, \phi^t u). \quad \square\end{aligned}$$

If K, L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for real $p > 0$ define the L_p-mixed volume of the bodies by:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u)\, dS_p(K, u). \tag{1.11}$$

¿From (1.11), (1.7), and the weak continuity of the classical surface area measures, it is easily seen that V_p is continuous in both arguments.

Suppose K, L are convex bodies that contain the origin in their interiors and real $\lambda, p > 0$. From definition (1.11), (1.7) and the fact that $h_{\lambda L} = \lambda h_L$, we have

$$V_p(K, \lambda L) = \lambda^p V_p(K, L) \qquad \text{and} \qquad V_p(\lambda K, L) = \lambda^{n-p} V_p(K, L). \tag{1.12}$$

An immediate consequence of Proposition 1.2 is:

Corollary 1.3. *If K, L are convex bodies that contain the origin in their interiors, real $p > 0$, and $\phi \in \mathrm{SL}(n)$, then*

$$V_p(\phi K, L) = V_p(K, \phi^{-1} L).$$

Proof. From definition (1.11), Proposition 1.2, Definition 1.1, the homogeneity of the support function, (1.1), and finally definition (1.11) again, we have

$$\begin{aligned}V_p(\phi K, L) &= \int_{S^{n-1}} h_L^p(u)\, dS_p(\phi K, u)\\ &= \int_{S^{n-1}} |\phi^{-t} u|^p h_L^p(\langle \phi^{-t} u\rangle)\, dS_p(K, u)\\ &= V_p(K, \phi^{-1} L). \quad \square\end{aligned}$$

Corollary 1.3, together with (1.11) and (1.12), shows that for $\phi \in \mathrm{GL}(n)$,

$$V_p(\phi K, \phi L) = |\phi| V_p(K, L), \tag{1.13}$$

where $|\phi|$ denotes the absolute value of the determinant of ϕ.

¿From (1.11) and Corollary 1.3, it follows that the original problem of minimizing total L_p-curvature under SL(n)-transformations can be rewritten as:

$$\begin{aligned}\min_{\phi\in \mathrm{SL}(n)} \int_{S^{n-1}} dS_p(\phi K, u) &= \min_{\phi\in \mathrm{SL}(n)} V_p(\phi K, B)\\ &= \min_{\phi\in \mathrm{SL}(n)} V_p(K, \phi^{-1}B)\\ &= \min_{|E|=\omega_n} V_p(K, E),\end{aligned}$$

where the last minimum is taken over all origin-centered ellipsoids whose volume is equal to that of the unit ball, B.

We shall make frequent use of the following formulation of *Jensen's inequality*: On a probability space the L_p means of a continuous function are strictly increasing in p, unless the function is constant. We shall also require the well-known fact that, as $p \to \infty$, the L_p means of the continuous function converge to its sup-norm.

In order to facilitate the formulation of our problem for the case $p = \infty$ it will be helpful to introduce a volume-normalized version of L_p mixed volumes. If K, L are convex bodies that contain the origin in their interiors, then for each real $p > 0$ define

$$\bar{V}_p(K,L) = \left(\frac{V_p(K,L)}{|K|}\right)^{\frac{1}{p}} = \left(\frac{1}{n|K|}\int_{S^{n-1}} \left[\frac{h_L(u)}{h_K(u)}\right)^p h_K(u)\, dS(K,u)\right]^{\frac{1}{p}}, \tag{1.14}$$

and for $p = \infty$ define

$$\bar{V}_\infty(K,L) = \max\{h_L(u)/h_K(u) : u \in \operatorname{supp} S(K,\ \cdot\ )\}. \tag{1.15}$$

Note that $\frac{1}{n} h_K\, dS(K,\ \cdot\ )/|K|$ is a probability measure on $\operatorname{supp} S(K,\ \cdot\ )$. Unless h_L/h_K is constant on $\operatorname{supp} S(K,\ \cdot\ )$, it follows from (1.14) and Jensen's inequality that

$$\bar{V}_p(K,L) < \bar{V}_q(K,L), \tag{1.16}$$

for $0 < p < q \le \infty$, and

$$\lim_{p\to\infty} \bar{V}_p(K,L) = \bar{V}_\infty(K,L).$$

We shall require the fact that, for $p_o \in (0,\infty]$,

$$\lim_{p\to p_o} \bar{V}_p(K,L) = \bar{V}_{p_o}(K,L). \tag{1.17}$$

¿From (1.12) and (1.14) it follows immediately that for $\lambda > 0$ and $p \in (0,\infty]$,

$$\bar{V}_p(\lambda K, L) = \lambda^{-1}\bar{V}_p(K,L) \qquad \text{and} \qquad \bar{V}_p(K,\lambda L) = \lambda \bar{V}_p(K,L). \tag{1.18}$$

We shall need the fact that for $\phi \in \mathrm{GL}(n)$ and all $p \in (0, \infty]$,

$$\bar{V}_p(\phi K, \phi L) = \bar{V}_p(K, L). \tag{1.19}$$

This follows immediately from (1.14) and (1.13) for real $p > 0$. But (1.19) for real $p > 0$ together with (1.17) shows that it in fact (1.19) holds for all $p \in (0, \infty]$.

Finally, we will require the fact that

$$\bar{V}_\infty(K, L) \leq 1 \qquad \text{if and only if} \qquad L \subseteq K. \tag{1.20}$$

This is a direct consequence of definition (1.15) and the fact that a convex body Q is the intersection of its supporting half spaces whose outer unit normals lie in the set $\operatorname{supp} S(Q, \,\cdot\,) \subseteq S^{n-1}$.

In order to establish the continuity of the L_p John ellipsoids in (Section 3) we shall require the crude estimate of the next Lemma. Throughout we shall use $|\cdot|_\infty$ to denote the sup norm on the space of continuous functions defined on $\operatorname{supp} S(K, \cdot) \subseteq S^{n-1}$.

Lemma 1.4. *If K, L, L_o are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then*

$$|\bar{V}_p(K, L) - \bar{V}_p(K, L_o)| \;\leq\; \frac{|h_L - h_{L_o}|_\infty}{\min_{u \in S^{n-1}} h_K(u)}, \tag{1.21}$$

for all $p \in [1, \infty]$.

Proof. First suppose $p < \infty$. From definition (1.14) together with the Minkowski inequality (i.e., the triangle inequality for L_p norms) we have:

$$\begin{aligned}
|\bar{V}_p(K, L) - \bar{V}_p(K, L_o)| &\leq \left[\frac{1}{n|K|}\int_{S^{n-1}} \left|\frac{h_L(u)}{h_K(u)} - \frac{h_{L_o}(u)}{h_K(u)}\right|^p h_K(u)\, dS(K, u)\right]^{1/p} \\
&\leq \left[\frac{1}{n|K|}\int_{S^{n-1}} \frac{1}{h_K(u)^p} h_K(u)\, dS(K, u)\right]^{1/p} |h_L - h_{L_o}|_\infty \\
&\leq \frac{|h_L - h_{L_o}|_\infty}{\min_{u \in S^{n-1}} h_K(u)}.
\end{aligned}$$

Taking the limit as $p \to \infty$, and using (1.17) shows that (1.21) holds for $p = \infty$ as well.

2. L_p John ellipsoids

Throughout, we assume that $p \in (0, \infty]$, and K is a convex body that contains the origin in its interior. As was the case above, E will always denote an origin-centered ellipsoid. We formulate our main problem in two equivalent ways.

Optimization Problems. *Given a convex body K in $\mathbb{R}^n$ that contains the origin in its interior, find an ellipsoid, amongst all origin-centered ellipsoids, which solves the following constrained maximization problem:*

$$\max(|E|/\omega_n)^{1/n} \qquad \textit{subject to} \qquad \bar{V}_p(K,E) \le 1. \tag{S_p}$$

A maximal ellipsoid will be called an S_p solution for K. The dual problem is:

$$\min \bar{V}_p(K,E) \qquad \textit{subject to} \qquad (|E|/\omega_n)^{1/n} \ge 1. \tag{$\bar{S}_p$}$$

A minimal ellipsoid will be called an $\bar{S}_p$ solution for K.

The solutions to S_p and $\bar{S}_p$ differ by only a scale factor.

Lemma 2.1. *Suppose $0 < p \le \infty$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. If E is an ellipsoid centered at the origin that is an $\bar{S}_p$ solution for K, then*

$$\left(\frac{|K|}{V_p(K,E)}\right)^{1/p} E \tag{2.1a}$$

is an S_p solution for K. If E' is an ellipsoid centered at the origin that is an S_p solution for K, then

$$(\omega_n/|E'|)^{1/n} E' \tag{2.1b}$$

is an $\bar{S}_p$ solution for K.

Given an ellipsoid E, with diameter $\operatorname{diam}(E)$, there exists a $v_E \in S^{n-1}$ such that $\operatorname{diam}(E)|v_E \cdot u|/2 \le h_E(u)$ for all $u \in S^{n-1}$. From this, definitions (1.11) and (1.14), and the constraint in (S_p) we see that

$$[\operatorname{diam}(E)/2]^p \min_{v \in S^{n-1}} \frac{1}{n} \int_{S^{n-1}} |v \cdot u|^p dS_p(K,u) \le V_p(K,E) \le |K|.$$

Thus, the diameters of a maximizing sequence of ellipsoids for Problem (S_p) are uniformly bounded and the existence of a solution for (S_p) is guaranteed by the Blaschke Selection Theorem. Lemma 2.1 now guarantees a solution to ($\bar{S}_p$) as well.

Theorem 2.2. *Suppose real $p > 0$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. Then S_p as well as $\bar{S}_p$ has a unique solution. Moreover, an ellipsoid E solves $\bar{S}_p$ if and only if it satisfies*

$$V_p(K,E) h^2_{E^*}(x) = \int_{S^{n-1}} |x \cdot v|^2 h_E^{p-2}(v)\, dS_p(K,v), \qquad \textit{for all } x \in \mathbb{R}^n; \tag{2.2a}$$

and an ellipsoid E solves S_p if and only if it satisfies

$$|K| h^2_{E^*}(x) = \int_{S^{n-1}} |x \cdot v|^2 h_E^{p-2}(v)\, dS_p(K,v), \qquad \textit{for all } x \in \mathbb{R}^n. \tag{2.2b}$$

By Lemma 2.1, only the assertions about an $\bar{S}_p$ solution require a proof. The existence of a solution has already been established, and only the uniqueness and the characterization statements require proof.

In order to establish Theorem 2.2, we first prove a lemma that shows that, without loss of generality, we may assume that the ellipsoid E is the unit ball, B, in $\mathbb{R}^n$.

Lemma 2.3. *Suppose real $p > 0$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. If $\phi \in \mathrm{GL}(n)$, then*

$$V_p(\phi^{-1}K, B)|x|^2 = \int_{S^{n-1}} |x\cdot v|^2 dS_p(\phi^{-1}K, v), \qquad \text{for all } x \in \mathbb{R}^n, \tag{2.3a}$$

if and only if

$$V_p(K, \phi B) h^2_{(\phi B)^*}(x) = \int_{S^{n-1}} |x\cdot v|^2 h^{p-2}_{\phi B}(v)\, dS_p(K, v), \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.3b}$$

Proof. In light of (1.12), it suffices to prove this for $\phi \in \mathrm{SL}(n)$. First note that

$$V_p(K, \phi B) h^2_{\phi^{-t}B^*}(x) = \int_{S^{n-1}} |x\cdot v|^2 h^{p-2}_{\phi B}(v)\, dS_p(K, v), \qquad \text{for all } x \in \mathbb{R}^n,$$

is by (1.1) and Corollary 1.3 equivalent to

$$V_p(\phi^{-1}K, B) h^2_{B^*}(\phi^{-1}x) = \int_{S^{n-1}} |x\cdot v|^2 h^{p-2}_{B}(\phi^t v)\, dS_p(K, v), \qquad \text{for all } x \in \mathbb{R}^n.$$

But using Definition 1.1 and Proposition 1.2 we see that this is equivalent to: For all $x \in \mathbb{R}^n$,

$$\begin{aligned} V_p(\phi^{-1}K, B) h^2_{B^*}(x) &= \int_{S^{n-1}} |\phi x\cdot v|^2 h^{p-2}_{B}(\phi^t v)\, dS_p(K, v) \\ &= \int_{S^{n-1}} |x\cdot \phi^t v|^2 |\phi^t v|^{p-2}\, dS_p(K, v) \\ &= \int_{S^{n-1}} |x\cdot \langle \phi^t v\rangle|^2 |\phi^t v|^{p}\, dS_p(K, v) \\ &= \int_{S^{n-1}} |x\cdot v|^2\, dS_p^{(p)}(K, \phi^{-t} v), \end{aligned}$$

which by Proposition 1.2 is in turn equivalent to

$$V_p(\phi^{-1}K, B)|x|^2 = \int_{S^{n-1}} |x\cdot v|^2 dS_p(\phi^{-1}K, v), \qquad \text{for all } x \in \mathbb{R}^n. \quad \square$$

Proof (of Theorem 2.2). We first show that if E is a $\bar{S}_p$ solution for K, then

$$V_p(K, E) h^2_{E^*}(x) = \int_{S^{n-1}} |x\cdot v|^2 h^{p-2}_{E}(v)\, dS_p(K, v), \qquad \text{for all } x \in \mathbb{R}^n.$$

Corollary 1.3 and Lemma 2.3 show that we may assume the $E = B$.

Suppose $l \in \mathrm{SL}(n)$. Choose $\varepsilon_0 > 0$ sufficiently small so that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ when εl is added to the identity matrix $1 \in \mathrm{SL}(n)$ the resulting matrix is invertible. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ define $l_\varepsilon \in \mathrm{SL}(n)$ by

$$l_\varepsilon = |1 + \varepsilon l|^{-1/n}(1 + \varepsilon l).$$

Since $|\, l_\varepsilon| = 1$, the ellipsoid $E_\varepsilon = l_\varepsilon^t B$ clearly has volume ω_n. The support function of E_ε is given by $h_{E_\varepsilon}(u) = h_{l_\varepsilon^t B}(u) = |\, l_\varepsilon u|$. The fact that B is an $\bar{S}_p$ solution, implies that $V_p(K, E_0) \le V_p(K, E_\varepsilon)$, for all ε, and hence using (1.11) we have:

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} |\, l_\varepsilon u|^p dS_p(K, u) = 0,$$

or equivalently,

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} |1 + \varepsilon l|^{-p/n}(u\cdot u + 2\varepsilon\, u\cdot lu + \varepsilon^2 lu\cdot lu)^{p/2} dS_p(K, u). \tag{2.3.1}$$

Since $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} |1 + \varepsilon l| = \mathrm{trace}(l)$ and since the integrand depends smoothly on ε (for small ε), from (2.3.1) we have

$$\int_{S^{n-1}} (u \cdot lu) dS_p(K, u) = V_p(K, B)\, \mathrm{trace}(l).$$

Choosing an appropriate l for each $i, j \in \{1, \dots, n\}$ gives

$$\int_{S^{n-1}} (u\cdot e_i)(u\cdot e_j) dS_p(K, u) = V_p(K, B)\delta_{ij},$$

which in turn gives

$$V_p(K, E)|x|^2 = \int_{S^{n-1}} |x\cdot v|^2 dS_p(K, v), \qquad \text{for all } x \in \mathbb{R}^n,$$

as desired.

Conversely, we suppose that

$$V_p(K, B) h_{B^*}^2(x) = \int_{S^{n-1}} |x\cdot v|^2 h_B^{p-2}(v)\, dS_p(K, v), \qquad \text{for all } x \in \mathbb{R}^n, \tag{2.3.2}$$

and shall prove that if $|E| = \omega_n$,

$$V_p(K, E) \ge V_p(K, B),$$

with equality if and only if $E = B$. Equivalently, we shall prove that if P is a positive definite symmetric matrix with $|P| = 1$, then

$$\left[\tfrac{1}{nV_p(K,B)} \int_{S^{n-1}} |Pu|^p dS_p(K, u)\right]^{\frac{1}{p}} \ge 1, \tag{2.3.3}$$

with equality if and only if $|Pu| = 1$ for all $u \in S^{n-1}$.

In order to establish (2.3.3) we shall prove:

$$\left[\tfrac{1}{nV_p(K,B)}\int_{S^{n-1}} |Pu|^p dS_p(K,u)\right]^{\frac{1}{p}} \geq \exp\left[\tfrac{1}{nV_p(K,B)}\int_{S^{n-1}} \log|Pu|\, dS_p(K,u)\right] \geq 1.$$

The left inequality is a direct consequence of Jensen's inequality with equality possible here if and only if there exists a $c > 0$ such that $|Pu| = c$ for all $u \in \operatorname{supp} S(K,\ \cdot\ )$.

Write P as $P = O^tDO$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and O is orthogonal. To establish our inequality we need show:

$$\int_{S^{n-1}} \log|Pu|\, dS_p(K,u) \geq 0. \tag{2.3.4}$$

First note that from (2.3.2) and Lemma 2.3 we get

$$V_p(OK,B)|x|^2 = \int_{S^{n-1}} |x\cdot v|^2 dS_p(OK,v), \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.3.5}$$

To get (2.3.4) note that from the fact that O is orthogonal, Definition 1.1, Proposition 1.2, the fact that D is diagonal, the concavity of the log function, and (2.3.5) we have

$$\begin{aligned}
\int_{S^{n-1}} \log|Pu|\, dS_p(K,u) &= \int_{S^{n-1}} \log|O^tDOu|\, dS_p(K,u) \\
&= \int_{S^{n-1}} \log|Du|\, dS_p(OK,u) \\
&\geq \frac{1}{2}\int_{S^{n-1}} (u_1^2 \log\lambda_1^2 + \cdots + u_n^2 \log\lambda_n^2) dS_p(OK,u) \\
&= V_p(OK,B)\sum_{i=1}^n \log\lambda_i \\
&= 0,
\end{aligned}$$

where we have abbreviated $u \cdot e_i$ by u_i.

Note that from the strict concavity of the log function it follows that equality in (2.3.4) is possible only if $u_{i_1}\cdots u_{i_N} \neq 0$ implies $\lambda_{i_1} = \cdots = \lambda_{i_N}$, for $u \in \operatorname{supp} S(OK,\ \cdot\ )$. Thus, $|Du| = \lambda_i$ when $u_i \neq 0$, for $u \in \operatorname{supp} S(OK,\ \cdot\ )$. Now equality in (2.3.3) would also force $|Pu| = c$ for all $u \in \operatorname{supp} S(K,\ \cdot\ )$, or equivalently $|Du| = c$ for all $u \in \operatorname{supp} S(OK,\ \cdot\ )$. Since $\operatorname{supp} S(OK,\ \cdot\ )$ is not contained in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, we have $\lambda_i = c$ for all i. This together with the fact that $\lambda_1 \cdots \lambda_n = 1$ shows that equality in (2.3.3) would imply $D = I$ and hence $P = I$. $\square$

Theorem 2.2 shows that problem (S_p) has a unique solution when $0 < p < \infty$. Now consider the case $p = \infty$ of (S_p). With the aid of (1.20), we may rephrase (S_∞) as: Amongst all origin-centered ellipsoids, find an ellipsoid which solves the following constrained maximization problem:

$$\max(|E|/\omega_n)^{1/n} \qquad \text{subject to} \qquad E \subseteq K. \tag{S_∞}$$

It is easily verified that a maximizing ellipsoid in (S_∞) is unique (see e.g. Giannopoulos and Milman [12]).

Definition 2.4. *Suppose K is a convex body that contains the origin in its interior and $0 < p \le \infty$. Amongst all origin-centered ellipsoids, the unique ellipsoid that solves the constrained maximization problem*

$$\max_E |E| \quad \textit{subject to} \quad \bar{V}_p(K, E) \le 1.$$

will be called the L_p John ellipsoid of K and will be denoted by $\mathrm{E}_p K$. Amongst all origin-centered ellipsoids, the unique ellipsoid that solves the constrained minimization problem

$$\min_E \bar{V}_p(K, E) \quad \textit{subject to} \quad |E| = \omega_n,$$

will be called the normalized L_p John ellipsoid of K and will be denoted by $\bar{\mathrm{E}}_p K$.

¿From (1.18) and (1.20) we immediately obtain:

Lemma 2.5. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $0 < p \le \infty$, then for $\phi \in \mathrm{GL}(n)$,*

$$\mathrm{E}_p \phi K = \phi\, \mathrm{E}_p K.$$

Obviously, $\mathrm{E}_p B = B$, and from Lemma 2.5 we see that if E is an ellipsoid that is centered at the origin, then

$$\mathrm{E}_p E = E. \tag{2.4}$$

Note that if the John point of K is at the origin (e.g., if K is origin-symmetric) then $\mathrm{E}_\infty K$ is the classical John ellipsoid of K.

¿From (0.5) and Theorem 2.2, we immediately obtain:

Lemma 2.6. *If K is a convex body that contains the origin in its interior, then*

$$\mathrm{E}_2 K = \Gamma_{-2} K.$$

3. Continuity

In this section we show that the family of L_p John ellipsoids associated with a convex body is continuous in $p \in (0, \infty]$.

Throughout this section K will be assumed to be a fixed convex body in $\mathbb{R}^n$ that contains the origin in its interior. For $u \in S^{n-1}$ let $\bar{u} = \{\lambda u : -1 \le \lambda \le 1\}$, and let $u^\perp$ denote the codimension 1 subspace of $\mathbb{R}^n$ that is orthogonal to u. Write $K|u^\perp$ for the image of the orthogonal projection of K onto $u^\perp$, and $\mathrm{vol}_{n-1}(K|u^\perp)$ for its $(n-1)$-dimensional volume.

Lemma 3.1. *Suppose $0 < p \le \infty$. If $aB \subseteq K \subseteq bB$, for $a, b > 0$, then $\bar{\mathrm{E}}_p K \subseteq c\,B$, for $c = (nb/a)^{\max\{1,1/p\}}$.*

Proof. Suppose $u \in S^{n-1}$. By writing $|K|$ as an integral over $K|u^\perp$ we see immediately that $|K|$ is bounded from above by $\operatorname{diam}(K)\operatorname{vol}_{n-1}(K|u^\perp)$. Thus

$$\frac{\operatorname{vol}_{n-1}(K|u^\perp)}{|K|} \;\geq\; \frac{1}{2b}. \tag{3.1.1}$$

Let $R_p\bar{u}_p$ denote the longest line segment contained in $\bar{\mathrm{E}}_pK$. Since $R_p\bar{u}_p \subset \bar{\mathrm{E}}_pK$,

$$R_p|u_p{\cdot}u| \;\leq\; h_{\bar{\mathrm{E}}_pK}(u), \tag{3.1.2}$$

for all $u \in S^{n-1}$.

¿From the Definition 2.6, (1.14) and (3.1.2), we have

$$\bar{V}_p(K,B) \geq \bar{V}_p(K,\bar{\mathrm{E}}_pK) \geq \left[\frac{1}{n|K|}\int_{S^{n-1}}\left(\frac{R_p|u_p{\cdot}u|}{h_K(u)}\right)^p h_K(u)\,dS(K,u)\right]^{1/p}. \tag{3.1.3}$$

Jensen's inequality and (3.1.3) show that when $p \geq 1$,

$$\bar{V}_p(K,B)/R_p \geq \frac{1}{n|K|}\int_{S^{n-1}}|u_p{\cdot}u|\,dS(K,u) = \frac{2\operatorname{vol}_{n-1}(K|u_p^\perp)}{n|K|}. \tag{3.1.4}$$

When $0 < p < 1$, from $aB \subseteq K$, we see that (3.1.3) gives

$$\begin{aligned}\bar{V}_p(K,B)/R_p &\geq \left[\frac{a^{1-p}}{n|K|}\int_{S^{n-1}}|u_p{\cdot}u|^p\,dS(K,u)\right]^{1/p}\\ &\geq \left[\frac{a^{1-p}}{n|K|}\int_{S^{n-1}}|u_p{\cdot}u|\,dS(K,u)\right]^{1/p}\\ &= \left[\frac{2a^{1-p}\operatorname{vol}_{n-1}(K|u_p^\perp)}{n|K|}\right]^{1/p}. \end{aligned}\tag{3.1.5}$$

Now (3.1.4) and (3.1.5), together with (3.1.1), give

$$\bar{V}_p(K,B) \geq \frac{R_p}{a}\left(\frac{a}{nb}\right)^{\max\{\frac{1}{p},1\}}. \tag{3.1.6}$$

To complete the proof we observe that by Jensen's inequality and (1.15)

$$\bar{V}_p(K,B) \leq \bar{V}_\infty(K,B) = \max\{1/h_K(u) : u \in \operatorname{supp} S(K,\;\cdot\;)\} \leq 1/a.$$

This and (3.1.6) yield the desired result. □

Given a $\delta > 0$, we see by Lemma 3.1 that there exists $R_{K,\delta} > 0$ such the compact set of ellipsoids,

$$\mathcal{E}_{K,\delta} = \{E : |E| = \omega_n \text{ and } E \subseteq R_{K,\delta}B\},$$

contains $\bar{\mathrm{E}}_p K$, and is independent of $p \in (\delta, \infty]$. Thus, in order to establish the continuity of $\bar{\mathrm{E}}_p K$ in $p \in (\delta, \infty]$ we may restrict the domain of $\bar{V}_p(K, \cdot)$ to $\mathcal{E}_{K,\delta}$.

Lemma 3.2. *If $p_o \in (\delta, \infty]$, then the limit*

$$\lim_{p \to p_o} \bar{V}_p(K, E) = \bar{V}_{p_o}(K, E),$$

is uniform for $E \in \mathcal{E}_{K,\delta}$.

Proof. Since $V_p(K, E)$ is a continuous function $(p, E) \in (\delta, 1] \times \mathcal{E}_{K,\delta}$, and $\mathcal{E}_{K,\delta}$ is compact, the functions $V_p(K, \cdot)$ with $p \in (\delta, 1]$ form an equicontinuous family of functions on $\mathcal{E}_{K,\delta}$. On the other hand, Lemma 1.4 shows that the $\bar{V}_p(K, \cdot)$, $p \in [1, \infty]$, also form an equicontinuous family of functions on $\mathcal{E}_{K,\delta}$. The lemma therefore follows by the Arzela-Ascoli theorem. $\square$

Recall that for each p, the ellipsoid $\bar{\mathrm{E}}_p K$ is the unique ellipsoid that satisfies:

$$\bar{V}_p(K, \bar{\mathrm{E}}_p K) = \min_{|E| = \omega_n} \bar{V}_p(K, E).$$

Lemma 3.3. *If $p_o \in (\delta, \infty]$, then*

$$\lim_{p \to p_o} \bar{V}_p(K, \bar{\mathrm{E}}_p K) = \bar{V}_{p_o}(K, \bar{\mathrm{E}}_{p_o} K).$$

Proof. Using the definition $\bar{\mathrm{E}}_p$, Lemma 3.2, (1.17), and again the definition of $\bar{E}_p$, we have

$$\begin{aligned}\lim_{p \to p_o} \bar{V}_p(K, \bar{\mathrm{E}}_p K) &= \lim_{p \to p_o} \min_{|E| = \omega_n} \bar{V}_p(K, E) \\ &= \min_{|E| = \omega_n} \lim_{p \to p_o} \bar{V}_p(K, E) \\ &= \min_{|E| = \omega_n} \bar{V}_{p_o}(K, E) \\ &= \bar{V}_{p_o}(K, \bar{\mathrm{E}}_{p_o} K). \quad \square\end{aligned}$$

Lemma 3.4. *If $p_o \in (\delta, \infty]$, then*

$$\lim_{p \to p_o} \bar{\mathrm{E}}_p K = \bar{\mathrm{E}}_{p_o} K.$$

Proof. We argue by contradiction and assume the conclusion to be false. Lemma 3.1, the Blaschke selection theorem, and our assumption, give a sequence $p_i \to p_o$, as $i \to \infty$, such that $\lim_{i \to \infty} \bar{\mathrm{E}}_{p_i} K = E' \neq \bar{\mathrm{E}}_{p_o} K$. Since the solution to Problem $(\bar{S}_p)$ is unique, and by the uniform convergence established in Lemma 3.2, we get

$$\begin{aligned}\bar{V}_{p_o}(K, \bar{\mathrm{E}}_{p_o} K) &< \bar{V}_{p_o}(K, E') \\ &= \lim_{i \to \infty} \bar{V}_{p_o}(K, \bar{\mathrm{E}}_{p_i} K) \\ &= \lim_{i \to \infty} \bar{V}_{p_i}(K, \bar{\mathrm{E}}_{p_i} K).\end{aligned}$$

This contradicts to Lemma 3.3. $\square$

Since $\delta > 0$ was arbitrary and since, by Lemma 2.1, $\mathrm{E}_p K = \bar{V}_p(K, \bar{\mathrm{E}}_p K)^{-1}\, \bar{\mathrm{E}}_p K$, the above gives:

Theorem 3.5. *Suppose K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. If $p_o \in (0, \infty]$, then*

$$\lim_{p \to p_o} \mathrm{E}_p K = \mathrm{E}_{p_o} K.$$

4. Generalizations of John's inclusion

John's inclusion states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\mathrm{E}_\infty K \subseteq K \subseteq \sqrt{n}\, \mathrm{E}_\infty K. \tag{4.1}$$

In this section, we shall prove an L_p version of John's inclusion.

If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and real $p > 0$, the star body $\Gamma_{-p} K$ is defined as the body whose radial function, for $u \in S^{n-1}$ is given by:

$$\rho_{\Gamma_{-p} K}(u)^{-p} = \frac{1}{|K|} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v). \tag{4.2}$$

Note for $p \geq 1$ the body $\Gamma_{-p} K$ is a convex body. Define $\Gamma_{-\infty} K$ by

$$\Gamma_{-\infty} K = \lim_{p \to \infty} \Gamma_{-p} K. \tag{4.3}$$

For real $p > 0$, use (1.7) and rewrite (4.2) as:

$$n^{-\frac{1}{p}} \rho_{\Gamma_{-p} K}(u)^{-1} = \left[\frac{1}{n|K|} \int_{S^{n-1}} \left(\frac{|u \cdot v|}{h_K(v)} \right)^p h_K(v)\, dS(K, v) \right]^{\frac{1}{p}},$$

for $u \in S^{n-1}$. Thus, from (4.3),

$$\rho_{\Gamma_{-\infty} K}(u)^{-1} = \max\{|u \cdot v| / h_K(v) : v \in \operatorname{supp} S(K, \ \cdot\ )\}.$$

Note that when K is origin-symmetric, $\Gamma_{-\infty} K = K$.

¿From (1.7) and definition (4.2) we see immediately that if $\lambda > 0$, then

$$\Gamma_{-p} \lambda K = \lambda \Gamma_{-p} K. \tag{4.4}$$

Lemma 4.1. *If $0 < p \leq \infty$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then for $\phi \in \mathrm{GL}(n)$,*

$$\Gamma_{-p} \phi K = \phi \Gamma_{-p} K.$$

Proof. From (4.4) we see that it is sufficient to prove the lemma when $\phi \in \mathrm{SL}(n)$. For real $p > 0$, it follows from definition (4.2), Proposition 1.2, Definition 1.1, and definition (4.2) again, that for $u \in S^{n-1}$:

$$\begin{aligned}
\rho_{\Gamma_{-p} \phi K}(u)^{-p} &= \frac{1}{|K|} \int_{S^{n-1}} |u \cdot v|^p\, dS_p^{(p)}(K, \phi^t v) \\
&= \frac{1}{|K|} \int_{S^{n-1}} |u \cdot \phi^{-t} v|^p\, dS_p(K, v) \\
&= \rho_{\Gamma_{-p} K}(\phi^{-1} u)^{-p}.
\end{aligned}$$

The L_∞ case is now a direct consequence of the real case and definition (4.3). $\square$

Lemma 4.2. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\begin{aligned} \mathrm{E}_p K \supseteq \Gamma_{-p} K \qquad &\textit{when} \qquad 0 < p < 2 \\ \mathrm{E}_p K \subseteq \Gamma_{-p} K \qquad &\textit{when} \qquad 2 < p \le \infty. \end{aligned}$$

Proof. Lemmas 2.5 and 4.1 show that it suffices to prove the inclusions when $\mathrm{E}_p K = B$.

For $0 < p < 2$, definition (4.2) and Theorem 2.2 show that for each $u \in S^{n-1}$,

$$\begin{aligned} \rho_{\Gamma_{-p}K}(u)^{-p} &= \frac{1}{|K|} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) \\ &\ge \frac{1}{|K|} \int_{S^{n-1}} |u \cdot v|^2 dS_p(K, v) \\ &= 1. \end{aligned}$$

This gives $\Gamma_{-p} K \subseteq B = \mathrm{E}_p K$ when $0 < p < 2$.

When $\infty > p > 2$, the inequality is reversed. Thus $\mathrm{E}_p K \subseteq \Gamma_{-p} K$ for real $p > 2$. The case $p = \infty$ follows from the real case together with Theorem 3.5 and definition (4.3). □

Of course the case $p = 2$ of Lemma 4.2 is known from Lemma 2.6: $\mathrm{E}_2 K = \Gamma_{-2} K$.

Our L_p version of John's inclusion will be a corollary of:

Theorem 4.3. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\begin{aligned} \Gamma_{-q} K \supseteq n^{\frac{1}{2} - \frac{1}{q}}\, \mathrm{E}_p K \qquad &\textit{when} \qquad 0 < q \le p \le 2 \\ \Gamma_{-q} K \subseteq n^{\frac{1}{2} - \frac{1}{q}}\, \mathrm{E}_p K \qquad &\textit{when} \qquad 2 \le p \le q \le \infty. \end{aligned}$$

Proof. Lemmas 2.5 and 4.1 show that it suffices to prove the inclusions when $\mathrm{E}_p K$ is the unit ball. Since $\mathrm{E}_p K = B$, Definition 2.4 gives

$$V_p(K, B) = |K|. \tag{4.3.1}$$

Suppose $0 < p < 2$. Now definition (4.2), definition (1.7), Jensen's inequality, definition 1.7 again, (4.3.1), Jensen's inequality again, (4.3.1) again, and finally Theorem

2.2, show that for each $u \in S^{n-1}$

$$\begin{aligned}
\rho_{\Gamma_{-q}K}(u)^{-1} &= n^{\frac{1}{q}} \left[\frac{1}{n|K|} \int_{S^{n-1}} \left(\frac{|u\cdot v|}{h_K(v)} \right)^q h_K(v)\, dS(K,v) \right]^{\frac{1}{q}} \\
&\le n^{\frac{1}{q}} \left[\frac{1}{n|K|} \int_{S^{n-1}} \left(\frac{|u\cdot v|}{h_K(v)} \right)^p h_K(v)\, dS(K,v) \right]^{\frac{1}{p}} \\
&= n^{\frac{1}{q}} \left[\frac{1}{nV_p(K,B)} \int_{S^{n-1}} |u\cdot v|^p dS_p(K,v) \right]^{\frac{1}{p}} \\
&\le n^{\frac{1}{q}} \left[\frac{1}{nV_p(K,B)} \int_{S^{n-1}} |u\cdot v|^2 dS_p(K,v) \right]^{\frac{1}{2}} \\
&= n^{\frac{1}{q}} \left[\frac{1}{n|K|} \int_{S^{n-1}} |u\cdot v|^2 dS_p(K,v) \right]^{\frac{1}{2}} \\
&= n^{\frac{1}{q}-\frac{1}{2}}.
\end{aligned}$$

Thus, $n^{\frac{1}{2}-\frac{1}{q}}\, \mathrm{E}_pK \subseteq \Gamma_{-q}K$.

When $\infty > q \ge p \ge 2$, the inequality above is reversed. Thus, $\Gamma_{-q}K \subseteq n^{\frac{1}{2}-\frac{1}{q}}\, \mathrm{E}_pK$ when $\infty > q \ge p \ge 2$. The case $q = \infty$ follows from the real case together with Theorem 3.5 and definition (4.3). $\square$

Recalling that for origin-symmetric K we have $\Gamma_{-\infty}K = K$, and choosing $q = \infty$ gives:

Corollary 4.4. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, and $2 \le p \le \infty$, then*

$$K \subseteq \sqrt{n}\, \mathrm{E}_pK.$$

By taking $p = q$ in Theorem 4.3 and combining the inclusions with those of Lemma 4.2 we get the L_p version of John's inclusion:

Corollary 4.5. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\begin{aligned}
&\mathrm{E}_pK \supseteq \Gamma_{-p}K \supseteq n^{\frac{1}{2}-\frac{1}{p}}\, \mathrm{E}_pK \qquad \text{when} \qquad 0 < p \le 2 \\
&\mathrm{E}_pK \subseteq \Gamma_{-p}K \subseteq n^{\frac{1}{2}-\frac{1}{p}}\, \mathrm{E}_pK \qquad \text{when} \qquad 2 \le p \le \infty.
\end{aligned}$$

5. Volume ratio inequalities

Theorem 5.1. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $0 < p < q \le \infty$, then*

$$|\,\mathrm{E}_qK| \le |\,\mathrm{E}_pK|.$$

Proof. From definitions (1.11) and (1.7), together with Jensen's inequality, it follows that for $0 < p \le q < \infty$,

$$\begin{aligned}\left[\frac{V_p(K,E)}{|K|}\right]^{\frac{1}{p}} &= \left[\frac{1}{n|K|}\int_{S^{n-1}}\left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u)dS(K,u)\right]^{\frac{1}{p}}\\ &\le \left[\frac{1}{n|K|}\int_{S^{n-1}}\left(\frac{h_E(u)}{h_K(u)}\right)^q h_K(u)dS(K,u)\right]^{\frac{1}{q}}\\ &= \left[\frac{V_q(K,E)}{|K|}\right]^{\frac{1}{q}}.\end{aligned}$$

The above together with Definition 2.4 immediately give the desired result for real q. For $q = \infty$ we use the real case together with Theorem 3.5. □

In general, the L_p John ellipsoid E_pK is not contained in K (except when $p = \infty$). However when $p \ge 1$, the volume of E_pK is always dominated by the volume of K:

Theorem 5.2. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $1 \le p \le \infty$, then*

$$|\,\mathrm{E}_pK| \le |K|,$$

with equality for $p > 1$, if and only if K is an ellipsoid centered at the origin, and equality for $p = 1$ if and only if K is an ellipsoid.

Proof. First suppose $p < \infty$. From Definition 2.4 and the L_p-Minkowski inequality (see [27]), we have

$$|K| = V_p(K, \mathrm{E}_pK) \ge |K|^{\frac{n-p}{n}}|\,\mathrm{E}_pK|^{\frac{p}{n}},$$

with equality for $p > 1$, if and only if $K = \mathrm{E}_pK$, and equality for $p = 1$ if and only if K and E_pK are translates. A glance at (2.4) now completes the proof when $p < \infty$. For $p = \infty$ combine this argument with Theorem 5.1 □

Theorem 5.1 and the Ball volume-ratio inequality (0.2), immediately give:

Theorem 5.3. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then for $0 < p \le \infty$,*

$$|K| \;\le\; \frac{2^n}{\omega_n}|\,\mathrm{E}_pK|,$$

with equality if and only if K is a parallelotope.

Note that if K is the cube $[-1,1]^n$, then $\mathrm{E}_pK = B$. This and Lemma 2.5 shows that for origin-centered parallelotopes there is indeed equality in the inequality of Theorem 5.3.

The case $p = \infty$ of Theorem 5.3 is due to Ball [3]. Our proof of Theorem 5.3 makes critical use of the Ball volume ratio inequality. A direct proof of the inequality can be found in [36] (see also [37]). The case $p = 2$ of Theorem 5.3 was established in [31].

6. Projections of convex bodies

If $p \in (0, \infty]$, and if K is an origin-symmetric convex body in $\mathbb{R}^n$, then K is said to be L_p *isotropic* if there exists a $c > 0$, such that

$$c|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \, dS_p(K, v), \qquad \text{for all } x \in \mathbb{R}^n.$$

Theorem 2.2 shows that K is L_p isotropic if and only if there exists a $\lambda > 0$, such that

$$\mathrm{E}_p K = \lambda B.$$

For $p = 1$, our next theorem was established by Giannopoulos and Papadimitrakis [13]. For $p = 2$ it was proved in [31]. Ball [1] had conjectured that the inequality of the theorem should hold for *some* affine transformation of the body.

Theorem 6.1. *If K is an origin-symmetric convex body in $\mathbb{R}^n$ that is L_p isotropic, for some $p \in [1, 2]$, then*

$$\mathrm{vol}_{n-1}(K|u^\perp) \leq \sqrt{n}|K|^{\frac{n-1}{n}}, \tag{6.1}$$

for all $u \in S^{n-1}$. There is equality in (6.1) for some $u \in S^{n-1}$ if and only if K is a cube and u is in the direction of one of the vertices.

Proof. If inequality (6.1) holds for a body K then it obviously holds for all dilates of the body. Thus we may assume that $\mathrm{E}_p K = B$.

¿From definition (4.2) and definition (1.7), together with Jensen's inequality we have

$$\begin{aligned} n^{-\frac{1}{p}} \rho_{\Gamma_{-p}K}(u)^{-1} &= \left[\frac{1}{n|K|} \int_{S^{n-1}} \left(\frac{|u \cdot v|}{h_K(v)} \right)^p h_K(v) dS(K, v) \right]^{\frac{1}{p}} \\ &\geq \frac{1}{n|K|} \int_{S^{n-1}} |u \cdot v| dS(K, v) \\ &= \frac{2}{n} \mathrm{vol}_{n-1}(K|u^\perp)/|K|. \end{aligned}$$

Since $\mathrm{E}_p K = B$, Definition 2.4 combined with definition (1.14), give

$$V_p(K, B) = |K|. \tag{6.1.1}$$

Definition (4.2), (6.1.1), Jensen's inequality, (6.1.1) again, and finally Theorem 2.2 give

$$\begin{aligned} n^{-\frac{1}{p}} \rho_{\Gamma_{-p}K}(u)^{-1} &= \left[\frac{1}{nV_p(K, B)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) \right]^{\frac{1}{p}} \\ &\leq \left[\frac{1}{nV_p(K, B)} \int_{S^{n-1}} |u \cdot v|^2 dS_p(K, v) \right]^{\frac{1}{2}} \\ &= \left[\frac{1}{n|K|} \int_{S^{n-1}} |u \cdot v|^2 dS_p(K, v) \right]^{\frac{1}{2}} \\ &= 1/\sqrt{n}, \end{aligned}$$

with equality if and only if there exists a $c > 0$ such that $|u \cdot v| = c$ for all $v \in \operatorname{supp} S(K, \cdot\,)$.

By combining the last two inequalities we have,

$$\operatorname{vol}_{n-1}(K|u^{\perp}) \le \sqrt{n}\,|K|/2, \tag{6.1.2}$$

with equality if and only if there exists a $c > 0$ such that $|u \cdot v| = c$ for all $v \in \operatorname{supp} S(K, \cdot\,)$.

¿From Theorem 5.3, we see that $\mathrm{E}_p K = B$ implies

$$|K|^{\frac{1}{n}} \le 2, \tag{6.1.3}$$

with equality if and only if K is a cube centered at the origin. Combining (6.1.2) and (6.1.3) gives the desired inequality.

If there is equality in (6.1) then K must be a cube centered at the origin. Since in this case $\operatorname{supp} S(K, \cdot\,) = \{\pm e_1, \dots, \pm e_n\}$, equality implies there exists a $c > 0$ such that $c = |u_i| = |u \cdot e_i|$ for all i, and hence $u = (\pm 1, \dots, \pm 1)/\sqrt{n}$. □

7. L_p-John ellipsoids of polar reciprocal bodies

If K is an origin symmetric convex body, the Blaschke-Sanatlaó inequality is the right side of

$$\frac{4^n}{n!} \le |K|\,|K^*| \le \omega_n^2.$$

There is equality in the right inequality if and only if K is an ellipsoid. The left inequality is a central conjecture, known as the Mahler conjecture: Among origin-symmetric convex bodies the *volume-product* is minimized by cubes and cross-polytopes (as well as other bodies). The left inequality has been verified for the class of zonoids (and their polars) by Reisner [41], [42] (see also [14]). For origin symmetric bodies, in general, the best results in the direction of the Mahler conjecture is the Bourgain-Milman inequality [6], which provides the correct asymptotic lower bound.

For the volumes of the L_p-John ellipsoids of polar reciprocal convex bodies we have:

Theorem 7.1. *If $p \in [1, \infty]$ and K is an origin-symmetric convex body, then*

$$\omega_n^2\, n^{-n/2} \le |\,\mathrm{E}_p K|\,|\,\mathrm{E}_p K^*| \le \omega_n^2, \tag{7.1}$$

with equality in the right inequality if and only if K is an ellipsoid and equality in the left inequality if K is a cube or the octahedron.

Proof. An immediate consequence of John's inclusion (0.1) and the definition of the polar body is that $n^{-1/2}\,\mathrm{E}_\infty^* K \subset K^*$. This and the definition of E_∞ shows that

$$|\,\mathrm{E}_\infty^* K| \le n^{n/2} |\,\mathrm{E}_\infty K^*|. \tag{7.1.1}$$

But $|\,\mathrm{E}_\infty K|\,|\,\mathrm{E}_\infty^* K| = \omega_n^2$ and this when combined with (7.1.1) gives the left inequality of (7.1).

To obtain the right inequality, combine Theorem 5.2 with the Blaschke-Santaló inequality. □

It is tempting to conjecture that the extremal bodies for the left inequality in Theorem 7.1 will turn out to be exactly the same as the extremal bodies of Mahler's conjecture.

8. Two theorems of Lewis

We now show how basic results regarding L_p-John ellipsoids can be used to obtain two important theorems of Lewis [22], [23].

An origin-symmetric convex body K in $\mathbb{R}^n$ gives rise to a Banach norm $\|\cdot\|_K$ on $\mathbb{R}^n$, defined for $x \in \mathbb{R}^n$ by,

$$\|x\|_K = 1/\rho_K(x), \tag{8.1}$$

for which K is the unit ball (i.e., $K = \{x \in \mathbb{R}^n : \|x\|_K \le 1\}$). Conversely if $(\mathbb{R}^n, \|\cdot\|)$ is a normed space with unit ball K (i.e., $K = \{x \in \mathbb{R}^n : \|x\| \le 1\}$), it is easily seen that $\|\cdot\| = \|\cdot\|_K$.

Suppose $(\mathbb{R}^n, \|\cdot\|)$ is isometric to an n-dimensional subspace of L_p. Hence there exists a Borel measure μ on S^{n-1} such that

$$\|x\| = \left(\int_{S^{n-1}} |x\cdot v|^p \, d\mu(v)\right)^{1/p}, \tag{8.2}$$

for all $x \in \mathbb{R}^n$. Since we are not dealing with an $(n-1)$-dimensional subspace, μ cannot be concentrated on a great sub-sphere of S^{n-1}. We may assume the measure μ is even (i.e., takes on the same value on antipodal Borel sets), if necessary, by replacing the measure μ by the even measure μ^*, defined by $2\mu^*(\omega) = \mu(\omega) + \mu(-\omega)$, for each Borel $\omega \subset S^{n-1}$. The solution to the even L_p Minkowski problem (see [35] and [27]) guarantees the existence of an origin-symmetric convex body K such that $\frac{1}{|K|} dS_p(K, \cdot\,) = d\mu$. This together with (8.2), (4.2), and (8.1), shows that $\|\cdot\| = \|\cdot\|_{\Gamma_{-p}K}$. This gives:

Lemma 8.1. *Suppose $1 \le p \le \infty$. The Banach space $(\mathbb{R}^n, \|\cdot\|)$ is an n-dimensional subspace of L_p if and only if there exists an origin-symmetric convex body K such that*

$$\|\cdot\| = \|\cdot\|_{\Gamma_{-p}K}.$$

An important theorem of Lewis [22] is:

Theorem 8.2. *If ℓ is an n-dimensional subspace of L_p, then ℓ is isometric to the Banach space $(\mathbb{R}^n, \|\cdot\|)$ where the norm $\|\cdot\|$ can be represented by a Borel measure, μ, such that for all $x \in \mathbb{R}^n$*

$$\|x\| = \left(\int_{S^{n-1}} |x\cdot v|^p \, d\mu(v)\right)^{1/p}, \tag{8.2a}$$

and

$$|x| = \left(\int_{S^{n-1}} |x\cdot v|^2 \, d\mu(v)\right)^{1/2}. \tag{8.2b}$$

Proof. By Lemma 8.1 there exists an origin-symmetric convex body K such that $\ell = (\mathbb{R}^n, \|\cdot\|_{\Gamma_{-p}K})$. Choose a $\phi \in \mathrm{GL}(n)$ that transforms the ellipsoid E_pK into the unit ball B. Thus from Lemma 2.5,

$$\mathrm{E}_p\phi K = \phi\,\mathrm{E}_pK = B. \tag{8.2.1}$$

Since their unit balls are $\mathrm{GL}(n)$-images, $(\mathbb{R}^n, \|\cdot\|_{\phi\Gamma_{-p}K})$ is isometric to ℓ. To see that for $(\mathbb{R}^n, \|\cdot\|_{\phi\Gamma_{-p}K})$ there is a measure with the desired properties (8.2a) and (8.2b), define the Borel measure μ on S^{n-1} by

$$d\mu = \frac{1}{|\phi K|} dS_p(\phi K,\ \cdot\ ). \tag{8.2.2}$$

¿From (8.2.2) and definition (4.2) together with (8.2.1) we get the desired (8.2a). But (8.2.2) and Theorem 2.2b together with (8.2.1) gives the desired property (8.2b). □

The Banach-Mazur distance, $d(X, X')$, between two n-dimensional Banach spaces $X = (\mathbb{R}^n, \|\cdot\|_K)$ and $X' = (\mathbb{R}^n, \|\cdot\|_{K'})$ is defined by

$$d(X, X') = \inf\{c > 0 : K \subseteq \phi K' \subseteq cK \text{ for some } \phi \in \mathrm{GL}(n)\}.$$

Corollary 4.5, together with Lemma 8.1, gives the following fundamental result of Lewis [22], [23] regarding the Banach-Mazur distance between an arbitrary n-dimensional subspace of L_p and ordinary Euclidean n-space, $\ell_2^n = (\mathbb{R}^n, \|\cdot\|_B)$:

Theorem 8.3. *Suppose $1 \le p < \infty$. If X is an n-dimensional Banach subspace of L_p, then*

$$d(X, \ell_2^n) \le n^{|\frac{1}{p} - \frac{1}{2}|}.$$

9. An open problem regarding the conical-volume measure of a body

Thus far L_p-John ellipsoids have been defined for $0 < p \le \infty$. A critical case is $p = 0$. This case is especially interesting because the 0-surface area measure $dS_0(K,\ \cdot\ ) = h_K dS(K,\ \cdot\ )$ is the *conical-volume measure* of K, which can be described as follows.

Recall that $u \in S^{n-1}$ is an *outer unit normal* at $x \in \partial K$, if it is the outer unit normal of some supporting hyperplane at x. Given $\omega \subseteq S^{n-1}$, let $C_K(\omega) \subset \partial K$ denote the set of all points that have an outer unit normal in ω. Now $S_0(K, \omega)$, the conical-volume measure of ω, is defined to be n times the volume of the cone obtained by taking the union of all line segments that connect the origin to a point in $C_K(\omega)$. The existence of E_0K can be formulated as follows:

Problem 9.1. *If K is a convex body in $\mathbb{R}^n$, that contains the origin in its interior, then does there exist a linear transformation $\phi \in \mathrm{GL}(n)$ so that*

$$|x|^2 = \int_{S^{n-1}} |x \cdot u|^2 dS_0(\phi K, u),$$

for all $x \in \mathbb{R}^n$?

In other words, Problem 9.1 asks if every convex body can be $\mathrm{SL}(n)$-transformed into a body that is L_0-isotropic. Under certain smoothness assumptions on ∂K, an affirmative answer to Problem 9.1 can be given by employing an argument similar to that used in establishing Theorem 2.3 . For arbitrary convex bodies, Problem 9.1 is interesting and appears to be non-trivial.

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