# MA-GY7033 Linear Algebra I

Homework 1

Throughout, $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$, and vector spaces are over $\mathbb{F}$ unless another field is specified. The symbol $\subseteq$ allows equality. The space $\mathcal{P}_n(\mathbb{F})$ consists of all polynomials of degree at most $n$ with coefficients in $\mathbb{F}$, including the zero polynomial. Operations on functions and sequences are pointwise unless stated otherwise.

Justify every answer. Use the definitions and results from `ma-gy7033lecture20260903.pdf`, together with elementary algebra and any results you have proved in earlier problems. Matrices, determinants, and theorems about linear maps are not needed. The final two problems are challenges.

## Axioms, subspaces, and span

**Problem 1. Consequences of the axioms**

Let $V$ be a vector space, $v \in V$, and $a \in \mathbb{F}$. Prove directly from the vector space axioms that:

(a) the zero vector and the additive inverse of each vector are unique;

(b) $0v = 0_V$ and $a0_V = 0_V$;

(c) $(-1)v = -v$;

(d) if $av = 0_V$, then $a = 0$ or $v = 0_V$.

Distinguish scalar zeros from vector zeros in your proofs.

**Problem 2. An unfamiliar vector space**

On $V = \{x \in \mathbb{R} : x > 0\}$, define

$$x \oplus y = xy, \qquad a \odot x = x^a \quad (a \in \mathbb{R}).$$

(a) Prove that these operations make $V$ a vector space over $\mathbb{R}$. Identify its zero vector and the additive inverse of $x$.

(b) Determine the dimension of $V$ and characterize all its bases.

You may use the usual properties of real powers and logarithms.

**Problem 3. What span does**

Let $S, T \subseteq V$ be arbitrary subsets, possibly empty or infinite. Recall that an empty linear combination is equal to $0_V$.

(a) Prove that $\operatorname{span}(S)$ is a subspace containing $S$.

(b) Prove that every subspace containing $S$ contains $\operatorname{span}(S)$. Deduce that $\operatorname{span}(S)$ is the intersection of all subspaces containing $S$.

**Problem 4. A parameter that changes the span**

For $a \in \mathbb{F}$, set

$$v_1 = (1, 0, 1), \qquad v_2 = (0, 1, 1), \qquad v_3 = (1, 1, a).$$

(a) Determine exactly when $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{F}^3$. For each remaining value of $a$, find a basis and the dimension of its span, and describe the span by a linear equation.

(b) When the three vectors form a basis, express an arbitrary $(x, y, z) \in \mathbb{F}^3$ as their linear combination.

**Problem 5. Solutions and differences of solutions**

In $\mathbb{F}^3$, define

$$H = \{(x, y, z) : x + y + z = 0,\ x - y = 0\},$$
$$A = \{(x, y, z) : x + y + z = 1,\ x - y = 0\}.$$

(a) Prove that $H$ is a subspace and find a basis and its dimension. Determine whether $A$ is a subspace.

(b) Find $p \in A$ for which $A = \{p + h : h \in H\}$, and prove this equality. Prove that the difference of any two elements of $A$ belongs to $H$.

## Polynomials, functions, and sequences

**Problem 6. A polynomial determined by three values**

Work in $\mathcal{P}_2(\mathbb{F})$.

(a) Construct $q_{-1}, q_0, q_1$ such that, for $j, k \in \{-1, 0, 1\}$,

$$q_j(k) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

(b) Prove that these polynomials form a basis and that every $p \in \mathcal{P}_2(\mathbb{F})$ satisfies

$$p = p(-1)q_{-1} + p(0)q_0 + p(1)q_1.$$

(c) Find the unique polynomial $p \in \mathcal{P}_2(\mathbb{F})$ for which $p(-1) = 2$, $p(0) = -1$, and $p(1) = 4$.

**Problem 7. A vector space of recursively defined sequences**

Let $V$ be the set of infinite sequences $(x_0, x_1, \ldots)$, where each $x_k \in \mathbb{F}$, that satisfy

$$x_{n+2} = x_{n+1} + x_n \qquad (n \geq 0).$$

(a) Prove that $V$ is a vector space where vector addition is

$$(x_0, x_1, \ldots) + (y_0, y_1, \ldots) = (x_0 + y_0, x_1 + y_1, \ldots)$$

and scalar multiplication is

$$s(x_0, x_1, \ldots) = (sx_0, sx_1, \ldots).$$

Construct a basis and determine its dimension.

(b) Prove that the only sequence in $V$ with finitely many nonzero terms is the zero sequence.

*Hint.* For (a), consider the two sequences with initial values $(x_0, x_1) = (1, 0)$ and $(x_0, x_1) = (0, 1)$.

## Bases and dimension

**Problem 8. Intersecting two spans**

In $\mathbb{F}^4$, let

$$U = \operatorname{span}\{(1,0,1,0), (0,1,0,1)\},$$
$$W = \operatorname{span}\{(1,1,0,0), (0,0,1,1)\}.$$

(a) Find bases and dimensions of $U$, $W$, and $U \cap W$.

(b) Define $U + W = \{u + w : u \in U,\ w \in W\}$. Prove directly that $U + W$ is a subspace, find a basis and its dimension, and describe it by one linear equation. Do not assume a dimension formula for sums of subspaces.

**Problem 9. Extending an independent set**

Let $V$ have dimension $n$.

(a) Prove that every linearly independent subset of $V$ is finite and can be extended to a basis of $V$.

(b) Deduce that any $n$ linearly independent vectors form a basis, and that any set of $n$ vectors spanning $V$ is a basis.

(c) Prove that a subspace $W \subseteq V$ with $\dim W = n$ must equal $V$. Give an example showing that an infinite dimensional space can have a proper subspace of the same dimension; exhibit bases to justify the comparison.

*Hint.* For (a), show that adjoining $v \notin \operatorname{span}(S)$ to an independent set $S$ preserves independence. Use the comparison theorem from the lecture to bound the number of steps.

**Problem 10. Testing plausible assertions**

Decide whether each statement is true or false. Prove every true statement and give an explicit counterexample for every false one. Here $S, T$ are subsets of a vector space $V$.

(a) If $S$ and $T$ are linearly independent, then $S \cup T$ is linearly independent.

(b) If $S$ and $T$ are linearly independent, then $S \cap T$ is linearly independent.

(c) If $S$ and $T$ are finite and $\operatorname{span}(S) = \operatorname{span}(T)$, then $S$ and $T$ have the same number of elements.

(d) $\operatorname{span}(S \cap T) = \operatorname{span}(S) \cap \operatorname{span}(T)$.

(e) If $S \subsetneq T$ and $T$ is linearly independent, then $\operatorname{span}(S) \subsetneq \operatorname{span}(T)$.