MA-GY7033 Linear Algebra I

Homework 2

Problem 1. Given a matrix M, let M^j_k denote the entry in the j-th row and k-the column of M. Let $A \in \mathcal{M}_{r\times s}$ and $B \in \mathcal{M}_{s\times t}$. Show that the definition of matrix multiplication given in lecture and slides is the same as

$$(AB)^j_l = \sum_{k=1}^{s} A^j_k B^k_l, \qquad 1 \le j \le r,\ 1 \le l \le t.$$

Problem 2. Let A, B, C are scalar-valued matrices such that $(AB)C$ and $A(BC)$ are well defined Use the formula above for matrix multiplication to prove associativity, i.e., $(AB)C = A(BC)$.

Problem 3. For $X, Y \in \mathcal{M}_{k\times l}$, let $P, Q \in \mathcal{M}_{k+l\times k+l}$ be given by

$$P = \begin{bmatrix} I_{k\times k} & X \\ 0 & I_{l\times l} \end{bmatrix} \text{ and } Q = \begin{bmatrix} I_{k\times k} & Y \\ 0 & I_{l\times l} \end{bmatrix}.$$

(a) Compute PQ.

(b) Given X, find a matrix Y such that $PQ = QP = I$.

Problem 4. (a) Fix $s \ge 1$. Prove that if $J \in \mathcal{M}_{r\times r}$ satisfies $JB = B$ for every $B \in \mathcal{M}_{r\times s}$, then $J = I_{r\times r}$.

(b) Show by example that, for every $r \ge 2$, $JB = B$ for a single nonzero B does not imply $J = I_{r\times r}$. What happens when $r = 1$?

Problem 5. Matrix multiplication defines a map

$$\begin{aligned} \mathcal{M}_{p\times q} \times \mathcal{M}_{q\times r} &\to \mathcal{M}_{p\times r} \\ (A, B) &\mapsto AB. \end{aligned}$$

This problem shows that matrix multiplication is a bilinear map.

(a) Show that given $A \in \mathcal{M}_{p\times q}$, the map

$$\begin{aligned} \mathcal{M}_{q\times r} &\to \mathcal{M}_{p\times r} \\ B &\mapsto AB \end{aligned}$$

is linear.

(b) Show that given $B \in \mathcal{M}_{p\times q}$, the map

$$\begin{aligned} \mathcal{M}_{p\times q} &\to \mathcal{M}_{p\times r} \\ A &\mapsto AB \end{aligned}$$

is linear.

Problem 6. Recall that matrix multiplication is not necessarily commutative.

(a) For $A \in \mathcal{M}_{r\times s}$ and $B \in \mathcal{M}_{p\times q}$, determine exactly when AB and BA are both defined.

(b) Determine when AB and BA have the same size.

(c) Given $r \geq 2$, find matrices $A, B \in \mathcal{M}_{r\times r}$ such that $AB \neq BA$.

(d) Let $E^i_j \in \mathcal{M}_{r\times r}$ be the matrix whose entry in row i and column j is 1 and all of whose other entries are 0. Prove that $\{E^i_j : 1 \leq i, j \leq r\}$ is a basis of $\mathcal{M}_{r\times r}$ and that

$$E^i_j E^k_l = \delta^k_j E^i_l,$$

where $\delta^k_j = 1$ if $j = k$ and 0 otherwise.

(e) Prove that if $A \in \mathcal{M}_{r\times r}$ satisfies $AB = BA$ for every $B \in \mathcal{M}_{r\times r}$, then $A = cI_{r\times r}$ for some $c \in \mathbb{F}$.

(f) Determine all $A \in \mathcal{M}_{r\times r}$ that commute with every E^i_i, $1 \leq i \leq r$. Conclude that, for $r \geq 2$, commuting with every diagonal matrix is a strictly weaker condition than the one in (d).

Hint. Compare the entries of AE^i_j and $E^i_j A$.

Problem 7. There are nonzero matrices A and B such that $AB = 0$.

(a) Find all $A \in \mathcal{M}_{2\times 2}$ with $A^2 = 0$.

(b) Find $A, B \in \mathcal{M}_{2\times 2}$ with $AB = 0$ and $BA \neq 0$.

Problem 8.

Prove that the set of upper triangular matrices $\{A \in \mathcal{M}_{r\times r} : A^j_k = 0 \text{ whenever } j > k\}$ is a subspace of $\mathcal{M}_{r\times r}$, compute its dimension, and prove that it is closed under matrix multiplication.

Problem 9. Let

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

be the matrices on the slide *Examples of Matrix Multiplication*.

(a) Compute $AB - BA$.

(b) Show that $X \in \mathcal{M}_{2\times 2}$ satisfies $AX = XA$ if and only if $X = pI + qA$ for some $p, q \in \mathbb{F}$.

(c) Show that $X = 0$ is the only $X \in \mathcal{M}_{2\times 2}$ with $AX = XB$.

(d) Show that $A^2 = A + 14I$.

(e) Use this to write A^3 in the form $pA + qI$.

(f) Use this to find $N \in \mathcal{M}_{2\times 2}$ such that $AN = NA = I$.

Problem 10. Call $M \in \mathcal{M}_{m\times m}$ *invertible* if there is $N \in \mathcal{M}_{m\times m}$ with $MN = NM = I$. Prove that such an N is unique (so M^{-1} is well defined), and that if M, N are invertible then so is MN, with $(MN)^{-1} = N^{-1}M^{-1}$.

Problem 11. Let $A, B \in \mathcal{M}_{m\times m}$ with $AB = I$.

(a) Prove that if x is a column matrix such that $Bx = 0$, then $x = 0$.

(b) Deduce that the columns of B are distinct and form a linearly independent subset of $\mathbb{F}^m$ with m elements, and hence a basis. Conclude that every $y \in \mathbb{F}^m$ can be written as $y = Bx$.

(c) Prove that $BA = I$.

Problem 12.

$$2x - 3y = 5, \qquad x + y = -1, \qquad 3x + 5y = 0,$$

(a) Show that this system has no solution.

(b) Find matrices A, v, and b such that the system is equivalen to

$$Av = b.$$

(c) Find the set of all $b \in \mathbb{F}^3$ for which $Av = b$ has a solution and show that given such a b, the solution is unique