MA-GY7033 Linear Algebra I

Homework 3

Problem 1. Let $\mathcal{P}_n(\mathbb{R})$ be the vector space of all polynomials of degree at most n in the variable x with real coefficients, including the zero polynomial. A basis of $\mathcal{P}_n(\mathbb{R})$ is

$$E = \begin{bmatrix} 1 & x & \cdots & x^n \end{bmatrix}.$$

(a) Recall that differentiation is a linear map,

$$\begin{aligned} D : \mathcal{P}_3(\mathbb{R}) &\to \mathcal{P}_3(\mathbb{R}) \\ f &\mapsto f' \end{aligned}$$

Find the matrix M such that

$$D(E) = EM.$$

(b) Find the matrix N such that

$$D^2(E) = EN,$$

where $D^2 = D \circ D : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$.

Problem 2. Let $C(\mathbb{R})$ be the vector space of continuous real-valued functions with domain $\mathbb{R}$. Prove that it is not finite dimensional.

Problem 3. Let $E = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$ be a basis of V. Consider the linear map $L : V \to V$ such that

$$\begin{aligned} L(e_1) &= e_2 \\ L(e_2) &= e_3 \\ L(e_3) &= 0 \end{aligned}$$

Show that if M is the matrix where $L(E) = EM$, then $M, M^2 \neq 0$ but $M^3 = 0$.

Problem 4. Let $m > 0$ be an integer. Find a matrix M such that

$$M, M^2, \ldots, M^{m-1} \neq 0 \text{ but } M^m = 0.$$

Problem 5. Let $\{u, v, w\}$ be a basis of a real vector space V.

(a) Show that $\{u + v + w, v + w, w\}$ is also a basis.

(b) Is it possible that a subset $\{v_1, v_2, v_3\} \subset V$ is linearly dependent but the set $\{v_2 + v_3, v_3 + v_1, v_1 + v_2\}$ is linearly independent?

Problem 6. Let V, W be finite dimensional vector spaces, with $\dim V = m$ and $\dim W = n$, and let $L : V \to W$ be a linear map. Consider the sets

$$K = \{v \in V \ : \ L(v) = 0\} \subset V$$

and

$$L(V) = \{L(v) \ : \ v \in V\} \subset W.$$

Recall that the empty set is a basis of the zero subspace.

(a) Prove that K is a subspace of V.

(b) Prove that $L(V)$ is a subspace of W.

(c) Prove that there exists a row matrix basis $E = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$ of V such that the following hold:

$$\begin{aligned} \begin{bmatrix} L(e_1) & \cdots & L(e_{m-k}) \end{bmatrix} &\text{ is a basis of } L(V) \\ \begin{bmatrix} e_{m-k+1} & \cdots & e_m \end{bmatrix} &\text{ is a basis of } K. \end{aligned}$$

(d) Prove that

$$\dim(K) + \dim(L(V)) = \dim(V).$$

This is known as the rank-nullity theorem.

(e) Prove that there exist row matrix bases E of V and F of W such that $L(E) = FM$, where $r = m - k$ and

$$M = \left[\begin{array}{c|c} I_{r\times r} & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right].$$