

MA-GY7033 Linear Algebra I

Homework 5

Determinant of a matrix Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{F}^n$. Let D be the antisymmetric multilinear function on $\mathbb{F}^n$ such that

$$D(e_1, \dots, e_n) = 1.$$

Recall that if $c_1, \dots, c_n \in \mathbb{F}^n$ are the columns of a matrix M , then the determinant of M is

$$\det(M) = D(c_1, \dots, c_n).$$

Computation of determinant Each column of M can be written in terms of the standard basis,

$$c_k = e_1 M_k^1 + \dots + e_n M_k^n.$$

The determinant can be computed by applying the antisymmetric and multilinear properties of D to

$$\begin{aligned} \det(M) &= D(c_1, \dots, c_n) \\ &= D(e_1 M_1^1 + \dots + e_n M_1^n, \dots, e_1 M_n^1 + \dots + e_n M_n^n). \end{aligned}$$

Problem 1. Let $A: V \times V \rightarrow \mathbb{R}$ be a bilinear function on a real vector space V such that $A(v, v) = 0$ for every $v \in V$. Show A is antisymmetric.

Problem 2. Show that

$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc.$$

Problem 3. How is $\det(-A)$ related to $\det A$ for an $n \times n$ matrix A ? Explain.

Problem 4. Show that

$$\det \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = adf. \tag{1}$$

Problem 5. Explain why the determinant of an upper triangular $m \times m$ matrix equals the product of its diagonal entries. In other words, explain why

$$\det \begin{bmatrix} A_1^1 & A_2^1 & \dots & A_{m-1}^1 & A_m^1 \\ 0 & A_2^2 & \dots & A_{m-1}^2 & A_m^2 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & A_m^m \end{bmatrix} = A_1^1 A_2^2 \dots A_m^m.$$

A detailed proof is not required; describe the process that gives this conclusion.

Problem 6. Let the columns of M be $c_1, \dots, c_n$. Prove that, given $j \neq k$ and $a \in \mathbb{F}$, if M' is the matrix whose columns are given by

$$\begin{aligned} c'_j &= c_j + ac_k \\ c'_i &= c_i \text{ if } i \neq j \end{aligned}$$

then

$$\det(M') = \det(M).$$

Problem 7. Use the computation procedure described above to compute

$$\det \left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \right).$$

Problem 8. Given a permutation $\sigma \in S_n$, the **permutation matrix** P_σ is the matrix whose k -th column is $c_k = e_{\sigma(k)}$, i.e.,

$$P_\sigma = [e_{\sigma(1)} \mid \cdots \mid e_{\sigma(n)}].$$

For example, $n = 3$ and σ is the permutation given by

$$\begin{aligned} \sigma(1) &= 2 \\ \sigma(2) &= 1 \\ \sigma(3) &= 3, \end{aligned}$$

then

$$P_\sigma = [e_{\sigma(1)} \mid e_{\sigma(2)} \mid e_{\sigma(3)}] = [e_2 \mid e_1 \mid e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Show that a matrix is a permutation matrix if and only if every entry is 0 or 1 and there is exactly one 1 in each row and each column.
- (b) Show that if

$$M = [c_1 \mid \cdots \mid c_n],$$

then

$$MP_\sigma = [c_{\sigma(1)} \mid \cdots \mid c_{\sigma(n)}],$$

- (c) Given a permutation σ and $v = (v^1, \dots, v^n) \in \mathbb{R}^n$, describe the linear map defined by P_σ . In other words, what is $P_\sigma v$?
- (d) Show that, for any two permutations σ and τ ,

$$P_{\sigma \circ \tau} = P_\sigma P_\tau$$

by showing that for any $k \in \{1, \dots, n\}$,

$$P_{\sigma \circ \tau} e_k = P_\sigma P_\tau e_k.$$

- (e) Show that $\det(P_\sigma) = \epsilon(\sigma)$.