MA-GY7033 Linear Algebra I

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Outline

Course Web Site

- https://www.math.nyu.edu/~dy444/MA-GY7033Fall2026
- You are required to have read and consult regularly this web page and pages linked to it
- "I did not see ... on the web site" is never a valid excuse
- Follow instructions to sign up for
  - Gradescope
  - Ed Discussions

Grading

- Course grade
  - Homework: 0%
  - Weekly quizzes based on homework: 20%
  - Midterm: 30%
  - Final: 50%
  - Tweaks
- Overall grading policy
  - Partial credit for correct answer
  - Full credit if correct answer is correctly justified
  - Incorrect logic and calculations will be heavily penalized

Office Hours, Homework, Announcements

- Office hours
  - Thursdays 3-5pm, 2 Metrotech 864
- Announcements
  - Ed Discussion
  - Turn on email or text notifications

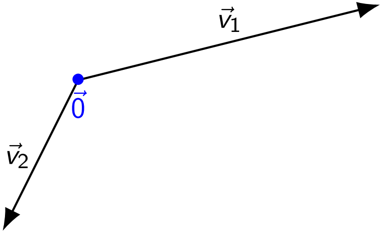
Overall Approach

- Linear algebra is a fundamental language and tool for
  - Geometry
  - Physics
  - Analysis
  - Any other field where linear functions are used
- Definitions and facts in linear algebra should not depend on units of length and distance

Vectors in the Plane

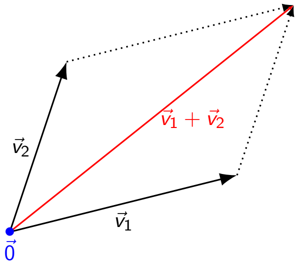
- There is a zero vector, also called the origin

- A vector is an arrow that starts at the origin and has a direction and magnitude (length)

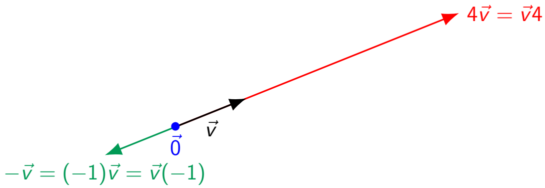


Vector Operations

- Vector addition



- Scalar multiplication



Properties of Vector Addition

- Notation

$$V \times V \to V$$
$$(\vec{v}_1, \vec{v}_2) \mapsto \vec{v}_1 + \vec{v}_2,$$

- Associativity

$$(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$$

- Commutativity

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$$

- Identity element: There exists an element $\vec{0} \in V$ such that, for any $\vec{v} \in V$,

$$\vec{v} + \vec{0} = \vec{v}$$

- Inverse element: For each $\vec{v} \in V$, there exists an element, written as $-\vec{v}$, such that

$$\vec{v} + (-\vec{v}) = \vec{0}$$

Properties of Scalar Multiplication

- Notation

$$
\begin{aligned}
\mathbb{R} \times V &\to V \\
(r, \vec{v}) &\mapsto r\vec{v} = \vec{v}r
\end{aligned}
$$

- Associativity

$$(r_1 r_2)\vec{v} = r_1(r_2\vec{v})$$

- Distributivity

$$(r_1 + r_2)\vec{v} = r_1\vec{v} + r_2\vec{v}$$
$$r(\vec{v}_1 + \vec{v}_2) = r\vec{v}_1 + r\vec{v}_2$$

- Identity element

$$1\vec{v} = \vec{v}$$

- Consequences

$$0\vec{v} = \vec{0}$$
$$(-1)\vec{v} = -\vec{v}$$

Another Fundamental Example: $\mathbb{R}^n$

- $\mathbb{R}^n = \{(v^1, \ldots, v^n) \ : \ v^1, \ldots, v^n \in \mathbb{R}\}$
- Fundamental operations
  - Vector addition
  - Scalar multiplication
- Same properties of vector addition and scalar multiplication as above
- The following operations are omitted for now:
  - Dot product
  - Cross product

Abstract Vector Space

- Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; an element of $\mathbb{F}$ is called a **scalar**
- An abstract vector space is a set of elements called vectors that have the same fundamental operations and properties as vectors in the plane
- A **vector space** over $\mathbb{F}$ is a set V with the following:
  - A special element, denoted $0_V \in V$, called the **zero vector**
  - An operation called vector addition:

$$\begin{aligned} V \times V &\to V \\ (v_1, v_2) &\mapsto v_1 + v_2 \end{aligned}$$

  - An operation called scalar multiplication:

$$\begin{aligned} V \times \mathbb{F} &\to V \\ (v, r) &\mapsto rv = vr \end{aligned}$$

- Observe that a vector space is always nonempty, because it always contains the zero vector

Properties of Vector Addition

- Notation

$$V \times V \to V$$
$$(v_1, v_2) \mapsto v_1 + v_2,$$

- Associativity

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

- Commutativity

$$v_1 + v_2 = v_2 + v_1$$

- Identity element: There exists an element $0_V \in V$ such that, for any $v \in V$,

$$v + 0_V = v$$

- Inverse element: For each $v \in V$, there exists an element, written as $-v$, such that

$$v + (-v) = 0_V$$

Properties of Scalar Multiplication

- Notation
$$\mathbb{F} \times V \to V$$
$$(r, v) \mapsto rv = vr$$
- Associativity
$$(r_1 r_2)v = r_1(r_2 v)$$
- Distributivity
$$(r_1 + r_2)v = r_1 v + r_2 v$$
$$r(v_1 + v_2) = rv_1 + rv_2$$
- Identity element
$$1v = v$$
- Consequences
$$0v = 0$$
$$(-1)v = -v$$

Examples

- The original example is $\mathbb{F}^n$, whose elements are of the form

$$v = \langle r^1, \ldots, r^n \rangle = \begin{bmatrix} r^1 \\ \vdots \\ r^n \end{bmatrix},$$

  where $r^1, \ldots, r^n \in \mathbb{F}$
- The set of all solutions to

$$3x - y + 2z = 0 \text{ and } x + 2y - z = 0$$

- Polynomials of degree 10 or less
- Polynomials of any degree
- Continuous functions $f : \mathbb{F} \to \mathbb{F}$

Why Abstract Vector Spaces Are Useful

- If we can prove something using only the properties of an abstract vector space, then it has to be true for any specific vector space
- Often, the statement and proof of a theorem are *easier* for an abstract space than for a specific example such as $\mathbb{R}^2$

Mathematical Grammar

- Invalid expressions
  - $a + b$, where $a \in \mathbb{R}$ and $b \in V$
  - ab, where $a, b \in V$
- When you write a formula or do a calculation,
  - Make sure you are adding or multiplying correctly
  - This is a good way to catch your mistakes

Linear Combination of Vectors

- Given a nonempty finite set of vectors $v_1, \dots, v_m \in V$ and scalars $r^1, \dots, r^m \in \mathbb{F}$, the vector

$$r^1 v_1 + \cdots + r^m v_m$$

  is called a **linear combination** of $v_1, \dots, v_m$
- A linear combination of the empty set is equal to 0
- Given a subset $S \subset V$, not necessarily finite, the **span** of S is the set of all possible linear combinations of vectors in S

$$\operatorname{span}(S) = \{r^1 v_1 + \cdots + r^m v_m \ : \ m \geq 0,\ r^1, \dots, r^m \in \mathbb{F} \text{ and } v_1, \dots, v_m \in S\}$$

- The span of the empty set is $\{0\}$
- A vector space V is called **finite dimensional** if there is a finite set S of vectors such that

$$\operatorname{span}(S) = V$$

Linearly Independent Set of Vectors

- A subset $S \subset V$ is **linearly independent** if for any finite subset $\{v_1, \dots, v_r\} \subset S$, any of the following equivalent statements holds:
  - If $a^1, \dots, a^r \in \mathbb{F}$ are scalars such that
    $$v_1 a^1 + \cdots + v_r a^r = 0,$$
    then $a^1 = \cdots = a^r = 0$
  - If $a^1, \dots, a^r, b^1, \dots, b^r \in \mathbb{F}$ are scalars such that
    $$v_1 a^1 + \cdots + v_r a^r = v_1 b^1 + \cdots + v_r b^r,$$
    then $a^1 = b^1, \dots, a^r = b^r$
  - The empty set is linearly independent

Linearly Dependent Set of Vectors

- A subset of vectors, $\{v_1, \ldots, v_r\}$, is **linearly dependent** if any of the following holds:
  - The set is not linearly independent
  - There exist scalars $a^1, \ldots, a^r$, where at least one is nonzero, such that
  $$v_1 a^1 + \cdots + v_r a^r = 0$$
  - The empty set is not linearly dependent, but the set $\{0\}$ is

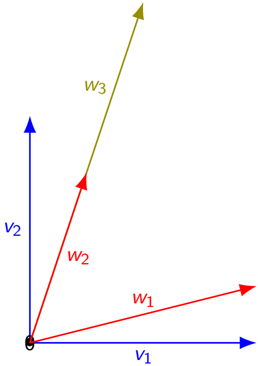
Examples of Linearly Independent and Linearly Dependent Sets

- $\{(1,0,0),(0,1,0)\} \subset \mathbb{F}^3$ is linearly independent
- $\{(1,0,0),(0,1,0),(1,1,0)\} \subset \mathbb{F}^3$ is linearly dependent
- The empty set is linearly independent
- $\{0\}$ is linearly dependent

Basis of a Vector Space

- A subset $S \subset V$ is a **basis** of V if it is a linearly independent set that spans V
- The only basis of the vector space $\{0\}$ is the empty set

Examples of Bases



- $\{v_1, v_2\}$ is a basis
- $\{w_1, w_2\}$ is a basis
- $\{w_1, w_3\}$ is a basis
- $\{w_2, w_3\}$ is NOT a basis

Examples of Bases

- Basis of the plane

$$S = \{(1,0),(0,1)\}$$
$$T = \{(2,1),(-1,1)\}$$

- Basis of $\mathbb{R}^3$

$$U = \{(1,0,0),(0,1,0),(0,0,1)\}$$
$$V = \{(2,1,0),(-1,1,0),(0,1,1)\}$$

- Not a basis of $\mathbb{R}^3$

$$W = \{(1,0,0),(0,1,0)\}$$
$$X = \{(2,1,0),(-1,1,1),(1,2,1)\}$$

Notation

- If $T \subset S$, then

$$S \backslash T = \{s \ : \ s \in S \text{ and } s \notin T\}$$

- $S \backslash T$ is called the complement of T in S

Linear Dependence Lemma (Axler, 2.21)

- If V is a vector space and $S = \{v_1, \ldots, v_k\} \subset V$ is linearly dependent, then there exists an element $v_j \in S$ such that

$$\text{span}(S \backslash \{v_j\}) = \text{span}(S)$$

- Proof
  - By the definition of linear dependence, there are scalars $s^1, \ldots, s^k$, not all zero, such that

$$s^1 v_1 + \cdots + s^k v_k = 0$$

  - One of the coefficients is nonzero and we can choose the order of the vectors such that $s^k \neq 0$
  - Therefore,

$$v_k = \frac{-s^1}{s^k} v_1 + \cdots + \frac{-s^{k-1}}{s^k} v_{k-1}$$

  - It follows that

$$v_k \in \text{span}(S \backslash \{v_k\})$$

    and therefore

$$\text{span}(S \backslash \{v_k\}) = \text{span}(S)$$

Every Finite Dimensional Vector Space Has a Basis

- Let V be a finite dimensional vector space
- Suppose $S_m = \{v_1, \ldots, v_m\} \subset V$ spans V
- If S_m is linearly independent, then S_m is a basis
- If not, then S_m is linearly dependent and, by the lemma above, there is a vector, call it $v_m \in S_m$ such that

  $$S_{m-1} = S_m \backslash \{v_m\} = \{v_1, \ldots, v_{m-1}\}$$

  spans V
- If S_{m-1} is linearly independent, it is a basis
- If not, repeat
- After repeating the steps above a finite number of times, the set S_k is linearly independent

Span of a Linearly Independent Set (Part 1)

- If $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_n\}$ are linearly independent sets in a vector space V such that

$$\operatorname{span}(S) \subset \operatorname{span}(T),$$

  then $m \le n$
- Proof
  - If $S = \emptyset$, then $m = 0 \le n$
  - Since $s_1 \in \operatorname{span}(T)$, there exist scalars $a^1, \ldots, a^n$, not all zero, such that

$$s_1 = a^1 t_1 + \cdots + a^n t_n$$

  - At least one of the coefficients is nonzero, say a^1, and therefore

$$t_1 = \frac{1}{a^1} s_1 + \frac{-a^2}{a^1} t_2 + \cdots + \frac{-a^n}{a^1} t_n$$

  - It follows that the set $T_1 = \{s_1, t_2, \cdots, t_n\}$ is linearly independent and $\operatorname{span}(T_1) = \operatorname{span}(T)$

Span of a Linearly Independent Set (Part 2)

- Suppose the set $T_j = \{s_1, \ldots, s_j, t_{j+1}, \ldots, t_n\}$ is linearly independent and $\text{span}(T_j) = \text{span}(T)$
  - We just proved this for $j = 1$
- If $j < m$, then since $s_{j+1} \in \text{span}(T)$, there are coefficients $a^1, \ldots, a^n$, not all zero (and not the same as the ones before), such that

$$s_{j+1} = a^1 s_1 + \cdots + a^j s_j + a^{j+1} t_{j+1} + \cdots + a^n t_n$$

- At least one of the $a^{j+1}, \ldots, a^n$ is nonzero, say a^{j+1}
- From this it follows that

$$t_{j+1} = \frac{1}{a^{j+1}} s_{j+1} + \frac{-a^1}{a^{j+1}} s_1 + \cdots + \frac{-a^j}{a^{j+1}} s_j + \frac{-a^{j+2}}{a^{j+1}} t_{j+2} + \cdots + \frac{-a^n}{a^{j+1}} t_n$$

- This implies that the set $T_{j+1} = \{s_1, \ldots, s_{j+1}, t_{j+2}, \ldots, t_n\}$ is linearly independent and $\text{span}(T_{j+1}) = \text{span}(T)$

Span of a Linearly Independent Set (Part 3)

- If $m > n$, then, after repeating this step n times, we find that

$$T_n = \{s_1, \ldots, s_n\} \subsetneq S$$

  is linearly independent and $\mathrm{span}(T_n) = \mathrm{span}(T)$
- This implies that

$$\mathrm{span}(T) = \mathrm{span}(T_n) \subsetneq \mathrm{span}(S) \subset \mathrm{span}(T),$$

  which is a contradiction
- Therefore, $m \leq n$
- It also follows that

$$m = n \iff \mathrm{span}(S) = \mathrm{span}(T)$$

Dimension of a Finite Dimensional Vector Space

- Corollary: Any two bases of V have the same number of elements
  - If S and T are bases of V, then $\text{span}(S) = \text{span}(T)$ and therefore $m = n$
- The dimension of a vector space is defined to be the number of elements in a basis
- A nonempty subset T of a vector space V is a **subspace** of V if for any $p, q \in \mathbb{F}$ and $a, b \in T$,

$$pa + qb \in T$$

- Every subspace T of V is itself a vector space using the same vector addition and scalar multiplication of V
- $\dim(T) \leq \dim(V)$ with equality holding if and only if $T = V$