MA-GY7033 Linear Algebra I

Systems of Linear Equations
Matrix Multiplication
Change of Basis Formulas
Linear Functions and Maps

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Outline

1. Matrix Multiplication
2. Change of Basis Formulas
3. Systems of Linear Equations
4. Linear Functions and Maps

Matrices

- A **column matrix** with r rows is of the form

$$A = \begin{bmatrix} a^1 \\ \vdots \\ a^r \end{bmatrix},$$

- A **row matrix** with s columns is of the form

$$B = \begin{bmatrix} b_1 & \cdots & b_s \end{bmatrix}$$

- A **matrix** with r rows and s columns is of the form

$$M = \begin{bmatrix} m_1^1 & \cdots & m_s^1 \\ \vdots & & \vdots \\ m_1^r & \cdots & m_s^r \end{bmatrix}$$

Product of a Row Matrix and a Column Matrix

- The matrix product of a row matrix B with s columns and a column matrix A with r rows is defined only when $r = s$:

$$\begin{aligned} BA &= \begin{bmatrix} b_1 & \cdots & b_s \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^s \end{bmatrix} \\ &= b_1 a^1 + \cdots + b_s a^s \end{aligned}$$

Product of a Column Matrix and a Row Matrix

- The matrix product of a column matrix A with r rows and a row matrix B with s columns is defined to be the r-by-s matrix

$$
\begin{aligned}
AB &= \begin{bmatrix} a^1 \\ \vdots \\ a^r \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_s \end{bmatrix} \\
&= \begin{bmatrix} a^1 b_1 & \cdots & a^1 b_s \\ \vdots & & \vdots \\ a^r b_1 & \cdots & a^r b_s \end{bmatrix}
\end{aligned}
$$

Generalized Matrix Multiplication

- The formulas for BA and AB are valid if any of the following hold:
  - B is a row matrix of scalars and A is a column matrix of scalars
    - Then BA is a scalar and AB is a scalar-valued matrix
  - B is a row matrix of vectors and A is a column matrix of scalars
    - Then BA is a vector and AB is a vector-valued matrix
  - B is a row matrix of scalars and A is a column matrix of vectors
    - Then BA is a vector and AB is a vector-valued matrix
- The formulas are invalid if B and A are both matrices of vectors

Generalized Matrices

- An r-by-s matrix is of the form

$$M = \begin{bmatrix} m_1^1 & \cdots & m_s^1 \\ \vdots & & \vdots \\ m_1^r & \cdots & m_s^r \end{bmatrix}$$

- M is a **matrix** or **scalar-valued matrix** if each m_k^j is a scalar
- M is a **vector-valued matrix** if each m_k^j is a vector
- M is a **matrix-valued matrix** if each m_k^j is a matrix

Matrix as Generalized Column Matrix

- M can be written as a column matrix of row matrices

$$M = \begin{bmatrix} R^1 \\ \vdots \\ R^r \end{bmatrix},$$

where, for each $1 \leq j \leq r$,

$$R^j = \begin{bmatrix} m_1^j & \cdots & m_s^j \end{bmatrix}$$

Matrix as Generalized Row Matrix

- M can be written as a row matrix of column matrices

$$M = \begin{bmatrix} C_1 & \cdots & C_s \end{bmatrix},$$

where, for each $1 \leq k \leq s$,

$$C_k = \begin{bmatrix} m_k^1 \\ \vdots \\ m_k^r \end{bmatrix}$$

Matrix Multiplication

- If A is an r-by-s matrix and B is an s-by-t matrix, then

$$A = \begin{bmatrix} R^1 \\ \vdots \\ R^r \end{bmatrix} \text{ and } B = \begin{bmatrix} C_1 & \cdots & C_t \end{bmatrix},$$

  where each R^j is a row matrix with s columns and each C_k is a column matrix with s rows

- The matrix product of A and B is defined to be the r-by-t matrix

$$AB = \begin{bmatrix} R^1 \\ \vdots \\ R^r \end{bmatrix} \begin{bmatrix} C_1 & \cdots & C_t \end{bmatrix} = \begin{bmatrix} R^1 C_1 & R^1 C_2 & \cdots & R^1 C_t \\ R^2 C_1 & R^2 C_2 & \cdots & R^2 C_t \\ \vdots & \vdots & \vdots & \vdots \\ R^r C_1 & R^r C_2 & \cdots & R^r C_t \end{bmatrix}$$

Scalar-Valued Matrices

- Most matrices will be scalar-valued matrices
- The set of all scalar-valued matrices with r rows and c columns will be denoted $\mathcal{M}_{r \times c}$
- The matrix in $\mathcal{M}_{r \times c}$ that has all zeros is called the **zero matrix** and will be denoted by $0_{r \times c}$ or simply 0
- Observe that $\mathcal{M}_{r \times c}$ is itself an abstract vector space, where the zero vector is the zero matrix
- The set of r-by-c vector-valued matrices, where the vectors are in a vector space V, will be denoted $\mathcal{M}_{r \times c}(V)$

The Identity Matrix

- Consider the square matrix $I_{r \times r} \in \mathcal{M}_{r \times r}$, where, for each $1 \leq j \leq r$, $I_j^j = 1$ and everything else is zero,

$$I_{r \times r} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Given any matrix $B \in \mathcal{M}_{r \times s}$,

$$I_{r \times r} B = B I_{s \times s} = B$$

- $I_{r \times r}$ is called the **identity matrix** and will usually be denoted by simply I

Properties of Matrix Multiplication

- Matrix multiplication works only if the number of columns in the first factor is equal to the number of rows in the second factor
  - If A is an r-by-s matrix and B is an s-by-t matrix, then AB is an r-by-t matrix
- Associativity: If A is an r-by-m matrix and B is an m-by-n matrix, C is an n-by-c matrix, where at least two of the matrices are scalar-valued, then

$$A(BC) = (AB)C$$

- Distributivity: If A is an r-by-m matrix and B, C are m-by-c matrices, then

$$A(B + C) = AB + AC$$

- $0_{q\times r}A = 0_{q\times m}$ and $A0_{m\times n} = 0_{r\times n}$
  - There exist nonzero matrices A and B such that $AB = 0_{r\times c}$
- $I_{r\times r}A = AI_{m\times m} = A$
- IMPORTANT: Matrix multiplication is NOT commutative

Examples of Matrix Multiplication

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2(0) + 3(1) & 2(1) + 3(-1) \\ 4(0) + (-1)(1) & 4(1) + (-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 0(2) + 1(4) & 0(3) + 1(-1) \\ 1(2) + (-1)4 & 1(3) + (-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}$$

Basis of Abstract Vector Space as Row Matrix

- A basis of a vector space $\mathbb{V}$ will be written as a row matrix of vectors:
$$E = (e_1, \ldots, e_m) = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$$
- For each $v \in \mathbb{V}$, there is a unique column matrix of scalars
$$a = \langle a^1, \ldots, a^m \rangle = \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}$$
such that
$$\begin{aligned} v &= e_1 a^1 + \cdots + e_m a^m \\ &= \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = Ea \end{aligned}$$

Change of Basis Matrix

- Let $E = (e_1, \dots, e_m)$ and $F = (f_1, \dots, f_m)$ be bases of V
- Since E is a basis, each $f_k \in F$ can be written as a linear combination of $e_1, \dots, e_m$,

$$f_k = e_1 M_k^1 + \cdots + e_m M_k^m$$

- Equivalently,

$$\begin{aligned} F &= \begin{bmatrix} f_1 & \cdots & f_m \end{bmatrix} \\ &= \begin{bmatrix} e_1 M_1^1 + \cdots + e_m M_1^m & \cdots & e_1 M_m^1 + \cdots + e_m M_m^m \end{bmatrix} \\ &= \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^m & \cdots & M_m^m \end{bmatrix} \\ &= EM \end{aligned}$$

Change of Basis Formula for a Vector

- A vector $v \in V$ can be written as a linear combination of the basis vectors $e_1, \ldots, e_n$,

$$v = e_1 a^1 + \cdots + e_m a^m = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = Ea,$$

- If $F = \begin{bmatrix} f_1 & \cdots & f_m \end{bmatrix}$ is also a basis, then there exist coefficients $b = (b^1, \ldots, b^m)$ such that

$$v = f_1 b^1 + \cdots + f_m b^m = Fb$$

- If $F = EM$ and $v = Ea = Fb$, it follows that

$$v = Ea \text{ but } v = Fb = (EM)b = E(Mb)$$

- Since the coefficients with respect to a basis are unique, it follows that

$$a = Mb \text{ i.e., } b = M^{-1}a$$

Change of Basis Matrices

- Assume
  - **The basis of each vector space is written as a row matrix**
  - **the coefficients of each vector is written as a column matrix**
- If M is matrix that changes the old basis into the new basis, then matrix that changes the coefficients of a vector with respect to the old basis to coefficients with respect to the new basis is M^{-1}
- **WARNING:** If the basis is written as a column matrix of vectors, then this is not true

Examples of Systems of Linear Equations

- The system

$$\begin{aligned} 2x - 3y &= 5 \\ x + y &= -1 \\ 3x + 5y &= 0 \end{aligned}$$

  can be written as

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

- The system

$$\begin{aligned} x + y + z &= 1 \\ 3y - 5z &= 2 \end{aligned}$$

  can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General System of Linear Equations

- A linear system

$$\begin{aligned} a^1_1 x^1 + \cdots + a^1_m x^m &= b^1 \\ &\vdots \\ a^n_1 x^1 + \cdots + a^n_m x^m &= b^n \end{aligned}$$

- Can be written using vectors and a matrix as

$$\begin{bmatrix} a^1_1 & \cdots & a^1_m \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_m \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} \text{or } Ax = b,$$

  where

$$x = \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix} \in \mathbb{F}^m,\ b = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} \in \mathbb{F}^n,\ A = \begin{bmatrix} a^1_1 & \cdots & a^1_m \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_m \end{bmatrix} \in \mathcal{M}_{n\times m}$$

- This is a system of n equations in m unknowns

Really Easy Examples

- 1 equation in 1 unknown (exactly one solution):

$$x = b \iff \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

- 1 equation in 2 unknowns (infinitely many solutions):

$$x = b \iff \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

- 2 equations in 1 unknown (Zero or one solutions)

$$\begin{aligned} x &= 1 \\ x &= 0 \end{aligned} \iff \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- 2 equations in 2 unknowns (Zero, exactly one, or infinitely many solutions):

$$\begin{aligned} x &= a \\ y &= b \end{aligned} \iff \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Linear Functions and Maps

- If V is a vector space, then a function $\ell : V \to \mathbb{F}$ is *linear*, if for any $v, v_1, v_2 \in V$ and $s \in \mathbb{F}$,

$$\begin{aligned} \ell(v_1 + v_2) &= \ell(v_1) + \ell(v_2) \\ \ell(sv) &= s\ell(v) \end{aligned}$$

- Easy to check that $\ell(0_V) = 0$
- If V and W are vector spaces, then a map $L : V \to W$ is *linear*, if for any $v, v_1, v_2 \in V$ and $s \in \mathbb{F}$,

$$\begin{aligned} L(v_1 + v_2) &= L(v_1) + L(v_2) \\ L(sv) &= sL(v) \end{aligned}$$

- Easy to check that $L(0_V) = 0_W$

Abstract Example of a Linear Map

- Let $(v_1, \ldots, v_n)$ be a basis of V
- Let $w_1, \ldots, w_n \in W$
- Define $L : V \to W$ as follows:
  - Set $L(v_i) = w_i$, for each $1 \leq i \leq n$
  - Since L is linear, it follows that given any

$$v = a^1 v_1 + \cdots + a^n v_n \in V,$$

the following must hold:

$$\begin{aligned} L(v) &= L(a^1 v_1 + \cdots + a^n v_n) \\ &= a^1 L(v_1) + \cdots + a^n L(v_n) \\ &= a^1 w_1 + \cdots + a^n w_n \end{aligned}$$

Concrete Example of a Linear Map

- Define a map $L : \mathbb{F}^2 \to \mathbb{F}^2$ as follows:

$$L((v^1, v^2)) = (3v^1 - 2v^2, 5v^1 + 6v^2)$$

- For example,

$$\begin{aligned} L(1,1) &= (3-2, 5+6) = (1, 11) \\ L(-2,3) &= (-6-6, -10+18) = (-12, 8) \end{aligned}$$

- Easy to check this is a linear map
- Using matrix multiplication, this can be written as

$$L\left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$