MA-GY7033 Linear Algebra I

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Outline

Examples of Systems of Linear Equations

- The system

$$\begin{aligned} 2x - 3y &= 5 \\ x + y &= -1 \\ 3x + 5y &= 0 \end{aligned}$$

  can be written as

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

- The system

$$\begin{aligned} x + y + z &= 1 \\ 3y - 5z &= 2 \end{aligned}$$

  can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General System of Linear Equations

- A linear system
$$\begin{aligned} a^1_1 x^1 + \cdots + a^1_m x^m &= b^1 \\ &\vdots \\ a^n_1 x^1 + \cdots + a^n_m x^m &= b^n \end{aligned}$$
- Can be written using vectors and a matrix as
$$\begin{bmatrix} a^1_1 & \cdots & a^1_m \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_m \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} \text{or } Ax = b,$$
where
$$x = \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix} \in \mathbb{F}^m, \; b = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} \in \mathbb{F}^n, \; A = \begin{bmatrix} a^1_1 & \cdots & a^1_m \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_m \end{bmatrix} \in \mathcal{M}_{n \times m}$$
- This is a system of n equations in m unknowns

Really Easy Examples

- 1 equation in 1 unknown (exactly one solution):

$$x = b \iff \begin{bmatrix}1\end{bmatrix}\begin{bmatrix}x\end{bmatrix} = \begin{bmatrix}b\end{bmatrix}$$

- 1 equation in 2 unknowns (infinitely many solutions):

$$x = b \iff \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}b\end{bmatrix}$$

- 2 equations in 1 unknown (Zero or one solutions)

$$\begin{aligned} x &= 1 \\ x &= 0 \end{aligned} \iff \begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}x\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

- 2 equations in 2 unknowns (Zero, exactly one, or infinitely many solutions):

$$\begin{aligned} x &= a \\ y &= b \end{aligned} \iff \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix}$$

Solving a Linear System (Part 1)

- Suppose we want to solve

$$2x - 4y + 6z = 12$$
$$3x + 12y + 24z = 36$$

- Rescale first equation to make x coefficient equal to 1

$$x - 2y + 3z = 6$$
$$3x + 12y + 24z = 36$$

- Subtract appropriate multiple of first equation from second equation to eliminate x from the second equation

$$x - 2y + 3z = 6$$
$$18y + 15z = 18$$

- Rescale second equation to make y coefficient equal to 1

$$x - 2y + 3z = 6$$
$$y + \frac{5}{6}z = 1$$

Solving a Linear System (Part 2)

- $$\begin{aligned} x - 2y + 3z &= 6 \\ y + \frac{5}{6}z &= 1 \end{aligned}$$
- Subtract appropriate multiple of second equation from first equation to eliminate y from the first equation
$$\begin{aligned} x + \frac{14}{3}z &= 8 \\ y + \frac{5}{6}z &= 1 \end{aligned}$$
- There are infinitely many solutions
$$\begin{aligned} x &= 8 - \frac{14}{3}z \\ y &= 1 - \frac{5}{6}z \\ z &= z \end{aligned}$$

Solving a Linear System Using the Matrix Equation

- $$\begin{array}{r} 2x - 4y + 6z = 12 \\ 3x + 12y + 24z = 36 \end{array} \iff \begin{array}{r} 2x - 4y + 6z - 12 = 0 \\ 3x + 12y + 24z - 36 = 0 \end{array}$$
  $$\iff \left[\begin{array}{ccc|c} 2 & -4 & 6 & 12 \\ 3 & 12 & 24 & 36 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
- Allowed operations
  - Switch order of equations $\iff$ Switch rows of matrix
  - Rescale coefficients of an equation $\iff$ Rescale a row of matrix
  - Add scalar multiple of one equation to another $\iff$ Add scalar multiple of one row to another
- The right side consists of elementary row operations

Example (Part 1)

$$\begin{aligned} x^1 + 2x^2 + 3x^3 + 5x^4 + x^5 &= 1 \\ x^3 + 5x^4 &= 2 \\ -3x^5 &= 9 \end{aligned}$$

$$\iff \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 5 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & -3 & 9 \end{array}\right] \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example (Part 2)

$$\begin{array}{c} \begin{array}{ccccc|c} x^1 & x^2 & x^3 & x^4 & x^5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 5 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & -3 & 9 \end{array}\right] \end{array} \to \begin{array}{c} \begin{array}{ccccc|c} x^1 & x^2 & x^3 & x^4 & x^5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 5 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right] \end{array}$$

$$\to \begin{array}{c} \begin{array}{ccccc|c} x^1 & x^2 & x^3 & x^4 & x^5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 5 & 0 & 4 \\ 0 & 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right] \end{array} \to \begin{array}{c} \begin{array}{ccccc|c} x^1 & x^2 & x^3 & x^4 & x^5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -10 & 0 & -2 \\ 0 & 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right] \end{array}$$

Example

- Left matrix is upper triangular

$$\begin{aligned} 2x + y + z &= 3 \\ y - 3z &= 4 \\ -z &= 2 \end{aligned} \implies \left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Work from bottom up:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right] \implies \left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\implies \left[\begin{array}{ccc|c} 2 & 1 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \implies \left[\begin{array}{ccc|c} 2 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{7}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \implies \begin{aligned} x &= \tfrac{7}{2} \\ y &= -2 \\ z &= -2 \end{aligned}$$

Solution Strategies

- Suppose you want to solve $Ax = b$, where A is square
- Above procedure is slow
- LU decomposition
  - Try to write $A = LU$, where L is lower triangular and U is upper triangular
  - First solve for y such that $Ly = b$
  - Then solve for x such that $Ux = y$
  - It follows that $LUx = Ly = b$
- Sparse matrix
  - A matrix is sparse if many of its entries are zero
  - Even better is if there is a pattern in where the zeros are
  - There are more efficient strategies to solving $Ax = b$ if A is sparse
  - See slides by Aleksandar Donev (Courant)

Normalization of a Linear System and Matrix

- Row operations
  - Switch two rows
  - Rescale a row
  - Add a scalar multiple of a row to another row
- Column operation
  - Linear system: Change order or names of variables
  - Matrix: Change order of columns

Possible Outcomes of Normalization 1

$$I_{m\times m}\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^m \end{bmatrix}$$

which is equivalent to

$$x^1 = b^1$$
$$\vdots$$
$$x^m = b^m$$

Possible Outcomes of Normalization 2

$$
\left[\begin{array}{c|c} I_{k\times k} & A_{k\times m-k} \end{array}\right]
\left[\begin{array}{c} x^1 \\ \vdots \\ x^k \\ \hline x^{k+1} \\ \vdots \\ x^m \end{array}\right]
=
\left[\begin{array}{c} b^1 \\ \vdots \\ b^k \end{array}\right],
$$

which is equivalent to

$$
\begin{aligned}
x^1 + a^1_{k+1}x^{k+1} + \cdots + a^1_m x^m &= b^1 \\
&\vdots \\
x^k + a^k_{k+1}x^{k+1} + \cdots + a^k_m x^m &= b^k
\end{aligned}
$$

Possible Outcomes of Normalization 3

$$
\left[\begin{array}{c} I_{k\times k} \\ \hline 0_{n-k\times k} \end{array}\right]
\begin{bmatrix} x^1 \\ \vdots \\ x^k \end{bmatrix}
=
\left[\begin{array}{c} b^1 \\ \vdots \\ b^k \\ \hline b^{k+1} \\ \vdots \\ b^n \end{array}\right],
$$

which is equivalent to

$$
\begin{aligned}
x^1 &= b^1 \\
&\vdots \\
x^k &= b^k \\
0 &= b^{k+1} \\
&\vdots \\
0 &= b^n
\end{aligned}
$$

Possible Outcomes of Normalization 4

$$\left[\begin{array}{c|c} I_{k\times k} & A_{k\times m-k} \\ \hline 0_{n-k\times k} & 0_{n-k\times m-k} \end{array}\right] \begin{bmatrix} x^1 \\ \vdots \\ x^k \\ \hline x^{k+1} \\ \vdots \\ x^m \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^k \\ \hline b^{k+1} \\ \vdots \\ b^n \end{bmatrix},$$

which is equivalent to

$$\begin{aligned} x^1 + a^1_{k+1}x^{k+1} + \cdots + a^1_m x^m &= b^1 \\ &\vdots \\ x^k + a^k_{k+1}x^{k+1} + \cdots + a^k_m x^m &= b^k \\ 0 &= b^{k+1} \\ &\vdots \\ 0 &= b^n \end{aligned}$$

Linear Functions and Maps

- If V is a vector space, then a function $\ell : V \to \mathbb{F}$ is *linear*, if for any $v, v_1, v_2 \in V$ and $s \in \mathbb{F}$,

$$\ell(v_1 + v_2) = \ell(v_1) + \ell(v_2)$$
$$\ell(sv) = s\ell(v)$$

- Easy to check that $\ell(0_V) = 0$
- If V and W are vector spaces, then a map $L : V \to W$ is *linear*, if for any $v, v_1, v_2 \in V$ and $s \in \mathbb{F}$,

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$
$$L(sv) = sL(v)$$

- Easy to check that $L(0_V) = 0_W$

Abstract Example of a Linear Map

- Let $(v_1, \ldots, v_n)$ be a basis of V
- Let $w_1, \ldots, w_n \in W$
- Define $L : V \to W$ as follows:
  - Set $L(v_i) = w_i$, for each $1 \leq i \leq n$
  - Since L is linear, it follows that given any

    $$v = a^1 v_1 + \cdots + a^n v_n \in V,$$

    the following must hold:

    $$\begin{aligned} L(v) &= L(a^1 v_1 + \cdots + a^n v_n) \\ &= a^1 L(v_1) + \cdots + a^n L(v_n) \\ &= a^1 w_1 + \cdots + a^n w_n \end{aligned}$$

Concrete Example of a Linear Map

- Define a map $L : \mathbb{F}^2 \to \mathbb{F}^2$ as follows:

$$L((v^1, v^2)) = (3v^1 - 2v^2, 5v^1 + 6v^2)$$

- For example,

$$\begin{aligned} L(1,1) &= (3-2, 5+6) = (1, 11) \\ L(-2,3) &= (-6-6, -10+18) = (-12, 8) \end{aligned}$$

- Easy to check this is a linear map
- Using matrix multiplication, this can be written as

$$L\left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

Differentiation is a Linear Map

- Let $C^0(\mathbb{R})$ be the space of functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous
- Let $C^1(\mathbb{R})$ be the space of functions $f : \mathbb{R} \to \mathbb{R}$ such that f and f' are continuous
- Both $C^0(\mathbb{R})$ and $C^1(\mathbb{R})$ are vector spaces
- The differentiation map is

$$\begin{aligned} D : C^1(\mathbb{R}) &\to C^0(\mathbb{R}) \\ f &\mapsto f' \end{aligned}$$

- By the sum and constant factor rules of differentiation,

$$\begin{aligned} D(f_1 + f_2) &= (f_1 + f_2)' \\ &= f_1' + f_2' \\ &= D(f_1) + D(f_2) \\ D(rf) &= (rf)' \\ &= rf' \\ &= rD(f) \end{aligned}$$

Linear Maps From $\mathbb{F}^2$ To $\mathbb{F}^3$

- Suppose $L : \mathbb{F}^2 \to \mathbb{F}^3$ is linear
- Suppose

$$L((1,0)) = (a^1, a^2, a^3)$$
$$L((0,1)) = (b^1, b^2, b^3)$$

- Given any vector $v = (v^1, v^2) \in \mathbb{F}^2$,

$$\begin{aligned} L(v) &= L(v^1(1,0) + v^2(0,1)) \\ &= v^1 L((1,0)) + v^2 L((0,1)) \\ &= v^1(a^1, a^2, a^3) + v^2(b^1, b^2, b^3) \\ &= \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \\ a^3 & b^3 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \end{aligned}$$

- Any linear map $L : \mathbb{F}^2 \to \mathbb{F}^3$ is given by a matrix $A \in \mathcal{M}_{3\times 2}$, where

$$L(v) = Av$$

Linear Maps from $\mathbb{F}^m$ to $\mathbb{F}^n$

- Consider a linear map $L : \mathbb{F}^m \to \mathbb{F}^n$
- Let $e_1, \ldots, e_m$ be the standard basis of $\mathbb{F}^m$
- For each $1 \leq k \leq m$, let $A_k = L(e_k) \in \mathbb{F}^n$
- For every $v = (v^1, \ldots, v^m) = v^1 e_1 + \cdots + v^m e_m \in \mathbb{F}^m$,

$$\begin{aligned} L(v) &= L(v^1 e_1 + \cdots + v^m e_m) \\ &= v^1 L(e_1) + \cdots + v^m L(e_m) \\ &= v^1 A_1 + \cdots + v^m A_m \\ &= \begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix} \\ &= Av \end{aligned}$$

Space of All Linear Maps from $\mathbb{F}^m$ to $\mathbb{F}^n$

- Let $\mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$ be the space of all linear maps from $\mathbb{F}^m$ to $\mathbb{F}^n$
- If $L_1, L_2 \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$, then $L_1 + L_2 \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$
- If $L \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$ and $r \in \mathbb{F}$, then $rL \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$
- Therefore, $\mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$ is a vector space
- There is a bijective linear map

$$\begin{aligned}\mathcal{L}(\mathbb{F}^m, \mathbb{F}^n) &\to \mathcal{M}_{n\times m}\\ L &\mapsto A, \text{ where, for any } v \in \mathbb{F}^m,\ L(v) = Av\end{aligned}$$

Linear Maps

- If V and W are vector spaces, then a map $L : V \to W$ is *linear*, if for any $v_1, v_2 \in V$ and $s^1, s^2 \in \mathbb{F}$,

$$L(s^1 v_1 + s^2 v_2) = s^1 L(v_1) + s^2 L(v_2)$$

- Example: $T : \mathbb{F}^2 \to \mathbb{F}^3$, where

$$T\left(\begin{bmatrix} a^1 \\ a^2 \end{bmatrix}\right) = \begin{bmatrix} 2a^1 - a^2 \\ a^1 + a^2 \\ -a^1 + 3a^2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Abstract Linear Map

- Given
  - Finite dimensional vector spaces V, W
  - A linear map $L : V \to W$
  - a basis $(e_1, \dots, e_m)$ of V
  - a basis $(f_1, \dots, f_n)$ be of W
- Each $L(e_k)$ is a linear combination of $(f_1, \dots, f_n)$:

$$
\begin{aligned}
L(e_1) &= M_1^1 f_1 + \cdots + M_1^n f_n \\
&\vdots \\
L(e_m) &= M_m^1 f_1 + \cdots + M_m^n f_n
\end{aligned}
$$

- This can be written as

$$
L\left(\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}\right) = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix}
$$

Abstract Linear Map With Respect To Bases (Part 1)

- Given
  - a basis $E = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$ of V
  - a basis $F = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}$ of W
  - a linear map $L : V \to W$
- There is a matrix

$$M = \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix}$$

  such that

$$L\left(\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}\right) = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix}$$

- I.e.,

$$L(E) = FM$$

Abstract Linear Map With Respect to Bases (Part 2)

- Given any vector

$$v = e_1 a^1 + \cdots + e_m a^m = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = Ea \in V,$$

$$\begin{aligned} L(v) &= L(e_1 a^1 + \cdots + e_m a^m) \\ &= L(e_1)a^1 + \cdots + L(e_m)a^m \\ &= (f_1 M_1^1 + \cdots + f_n M_1^n)a^1 + \cdots + (f_1 M_m^1 + \cdots + f_n M_m^n)a^m \\ &= f_1(M_1^1 a^1 + \cdots + M_m^1 a^m) + \cdots + f_n(M_1^n a^1 + \cdots + M_m^n a^m) \end{aligned}$$

- I.e.,

$$L(v) = L(Ea) = L(E)a = (FM)a = F(Ma)$$

Abstract Linear Map With Respect to Bases (Part 3)

- Upshot
  - Given
    - A linear map $L : V \to W$
    - A basis of V
    - A basis of W
  - There is a matrix that describes what the map L is
- If E is a basis of V and F is a basis of W, then there is a matrix M such that

$$L(E) = FM$$

- If $v = Ea$, then

$$L(v) = F(Ma)$$

Composition of Linear Maps

- Let $L_1 : V_0 \to V_1$ and $L_2 : V_1 \to V_2$ be linear maps
- The composition of L_2 and L_1 is the map

$$\begin{aligned} L_2 \circ L_1 : V_0 &\to V_2 \\ v_0 &\mapsto L_2(L_1(v_0)) \end{aligned}$$

- Let E_0 be a basis of V_0, E_1 a basis of V_1, E_2 a basis of V_2
- Let M_1 and M_2 be the matrices where

$$L_1(E_0) = E_1 M_1 \text{ and } L_2(E_1) = E_2 M_2$$

- It follows that

$$\begin{aligned} (L_2 \circ L_1)(E_0) &= L_2(L_1(E_0)) = L_2(E_1 M_1) = (L_2(E_1))M_1 \\ &= (E_2 M_2)M_1 = E_2(M_2 M_1) \end{aligned}$$

- And therefore, if $v = E_0 a \in V_0$, then

$$\begin{aligned} (L_2 \circ L_1)(v) &= (L_2 \circ L_1)(E_0 a) = (L_2 \circ L_1)(E_0)a \\ &= E_2(M_2 M_1)a = E_2(M_2 M_1 a) \end{aligned}$$

Inverse Map

- A linear map $L : V \to W$ is **invertible** if, for every $w \in W$, there is a unique $v \in V$ such that
$$L(v) = w$$
- Equivalently: L satisfies the following two properties:
  - L is onto: For any $w \in W$, there exists $v \in V$ such that
  $$L(v) = w$$
  - L is 1-to-1:
  $$L(v_1) = L(v_2) \implies v_1 = v_2$$
- Consequence: If
$$E = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \text{ is a basis of } V,$$
then
$$L(E) = \begin{bmatrix} L(e_1) & \cdots & L(e_m) \end{bmatrix} \text{ is a basis of } W$$
- This implies $\dim(V) = \dim(W)$

Matrix of Inverse Map

- If $L : V \to W$ is an invertible linear map, then there exists a map $K : W \to V$ such that $K \circ L = I_V$ and $L \circ K = I_W$
- (Prove:) K is a linear map
- We shall denote the map K by L^{-1}
- If E is a basis of V, F is a basis of W, and M, N are matrices such that
$$L(E) = FM \text{ and } L^{-1}(F) = EN,$$
then
$$F = I_W(F) = L(L^{-1}(F)) = L(EN) = L(E)N = FMN$$
$$E = I_V(E) = L^{-1}(L(E)) = L^{-1}(FM) = L^{-1}(F)M = ENM,$$
which implies that $MN = NM = I$
- We will denote N by M^{-1} and call it the inverse matrix of M

Change of Basis for Linear Map

- Let $L : V \to W$ be a linear map
- Given bases E of V and F of W, there is a matrix M such that
$$L(E) = FM$$
- Given bases E' of V and F' of W, there is a matrix M' such that
$$L(E') = F'M'$$
- Suppose $E' = EA$ and $F' = FB$ and therefore $E = E'A^{-1}$ and $F = F'B^{-1}$
- Then
$$L(E') = L(EA) = L(E)A = FMA = F'B^{-1}MA,$$
which implies that
$$M' = B^{-1}MA$$