

MA-GY7033 Linear Algebra I

Oriented Area and Volume
Antisymmetric Multilinear Functions
Determinant of Matrix

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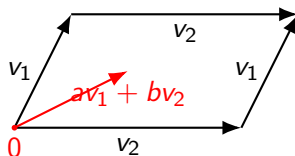
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Outline

1. Oriented Area and Volume
2. Antisymmetric Multilinear Functions
3. Determinant of Matrix

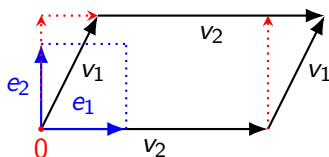
Parallelogram in Vector Space



- ▶ Let V be a 2-dimensional vector space
- ▶ Let $P(v_1, v_2)$ be the parallelogram with sides $v_1, v_2 \in V$.

$$P(v_1, v_2) = \{av_1 + bv_2 : 0 \leq a, b \leq 1\}.$$

Parallelogram With Respect To Basis



- ▶ With respect to basis (e_1, e_2)

$$v_1 = ae_1 + he_2 \text{ and } v_2 = we_1$$

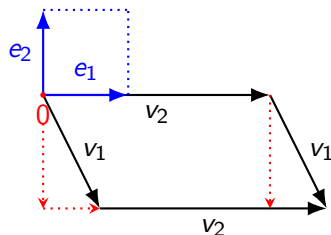
- ▶ Height is h and width is w
- ▶ Assume area of $P(e_1, e_2)$ is

$$A(e_1, e_2) = \text{area}(P(e_1, e_2)) = 1$$

- ▶ Then the area of $P(v_1, v_2)$ is

$$A(v_1, v_2) = \text{area}(P(v_1, v_2)) = hw$$

Upside Down Parallelogram With Respect To Basis



- ▶ With respect to basis (e_1, e_2)

$$v_1 = ae_1 + he_2 \text{ and } v_2 = we_1,$$

where h is negative

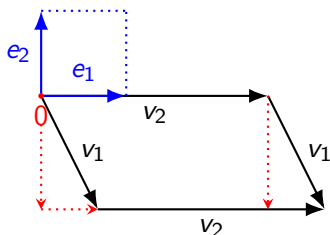
- ▶ Height is $|h|$ and width is w
- ▶ Then the area of $P(v_1, v_2)$ is

$$A(v_1, v_2) = |h|w,$$

whether h is positive or negative

- ▶ Formula is awkward due to absolute value

Oriented Area of Parallelogram

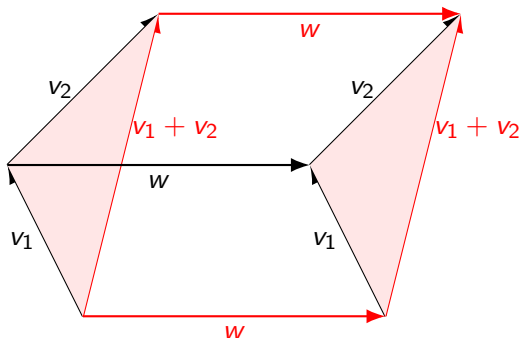


- Define oriented area to be

$$A(v_1, v_2) = -hw$$

- The oriented area of $P(v_1, v_2)$ is positive if v_2 lies counterclockwise of v_1
- The oriented area of $P(v_1, v_2)$ is negative if v_2 lies clockwise of v_1
- Oriented area, as a function of $v_1, v_2 \in V$ has nice properties

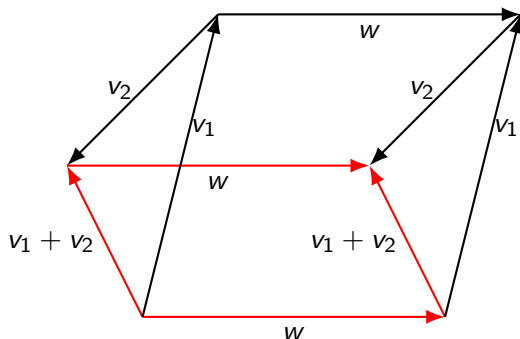
Area of Two Parallelograms with Parallel Bases



- If v_1 and v_2 both point upward relative to w , then

$$A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w)$$

Area of Two Parallelograms with Parallel Bases



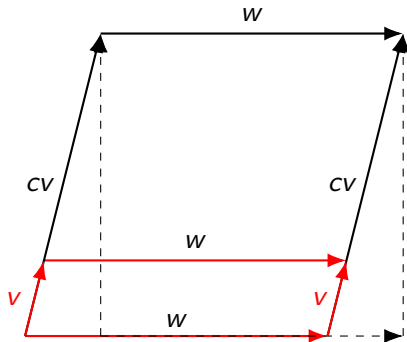
- ▶ If v_1 points upward and v_2 points downward relative to w , then $A(v_2, w) < 0$ and

$$A(v_1, w) = A(v_1 + v_2, w) - A(v_2, w)$$

and therefore

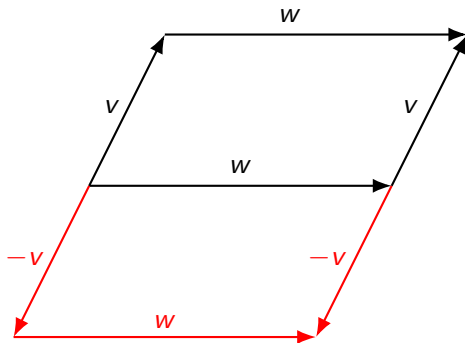
$$A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w)$$

Oriented area of rescaled parallelogram



$$A(cv, w) = cA(v, w)$$

Oriented area of reflected parallelogram



$$A(-v, w) = -A(v, w)$$

Area Versus Oriented Area

- ▶ Definitions of area and oriented area require a basis (e_1, e_2) , where we assume that

$$A(e_1, e_2) = 1$$

- ▶ In particular, e_2 must lie counterclockwise of e_1
- ▶ The area function $|A(v, w)|$ is awkward to use
- ▶ Instead, define $A(v, w)$ to be the *oriented area* of $P(v, w)$
- ▶ Define the oriented area of $P(v, w)$ to be

$$A(v, w) = \begin{cases} \text{area of } P(v, w) & \text{if } (v, w) \text{ is positively oriented} \\ -\text{area of } P(v, w) & \text{if } (v, w) \text{ is negatively oriented} \\ 0 & \text{if } v \text{ and } w \text{ are linearly dependent} \end{cases}$$

Oriented Area of Parallelogram

- ▶ If w is held fixed, $A(v, w)$ is a linear function of v

$$A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w)$$

$$A(cv, w) = cA(v, w)$$

- ▶ If v is held fixed, $A(v, w)$ is a linear function of w

$$A(v, w_1 + w_2) = A(v, w_1) + A(v, w_2)$$

$$A(v, cw) = cA(v, w)$$

- ▶ Such a function of two vectors is called **bilinear**
- ▶ For any $v \in V$, the parallelogram $A(v, v)$ has height 0 and therefore

$$A(v, v) = 0 \tag{1}$$

- ▶ Fact: Any bilinear function $A : V \times V \rightarrow \mathbb{R}$ that satisfies (1) is **antisymmetric**
- ▶ This means that for any $v, w \in V$,

$$A(w, v) = -A(v, w)$$

Determinant of a Matrix

- ▶ If $v = ae_1 + ce_2$ and $w = be_1 + de_2$, then

$$\begin{aligned}A(v, w) &= A(ae_1 + ce_2, be_1 + de_2) \\&= A(ae_1, be_1) + A(ce_2, be_1) + A(ae_1, de_2) + A(ce_2, de_2) \\&= bcA(e_2, e_1) + adA(e_1, e_2) \\&= ad - bc\end{aligned}$$

- ▶ This can be written as follows

$$\begin{aligned}A(v, w) &= A\left(E \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\&= A(ae_1 + ce_2, be_1 + de_2) \\&= A(e_1, e_2)(ad - bc) \\&= ad - bc\end{aligned}$$

- ▶ The determinant of a square 2-by-2 matrix is defined to be

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Determinant of a 2-by-2 Matrix is Equal to Oriented Area

- ▶ Let (e_1, e_2) be a basis where the oriented area of $P(e_1, e_2)$ is 1,

$$A(e_1, e_2) = 1$$

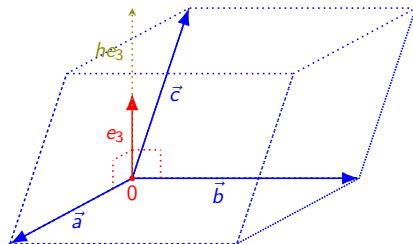
- ▶ If

$$[v \quad w] = [e_1 \quad e_2] \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [e_1 a + e_2 c \quad e_1 b + e_2 d]$$

then

$$A(v, w) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

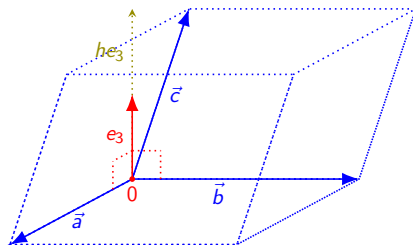
Parallelopiped spanned by 3 Vectors in 3-space



- ▶ Three linearly independent vectors $\vec{a}, \vec{b}, \vec{c}$ span a parallelopiped $P(\vec{a}, \vec{b}, \vec{c})$

$$P(\vec{a}, \vec{b}, \vec{c}) = \{s\vec{a} + t\vec{b} + u\vec{c} : 0 \leq s, t, u \leq 1\}$$

Volume of a Parallelepiped

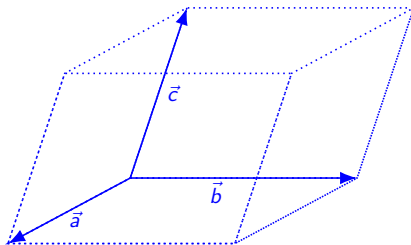


- ▶ Fix a basis (e_1, e_2, e_3) of V
 - ▶ Assume the volume of $P(e_1, e_2, e_3)$ is 1
- ▶ Assume $\vec{a}, \vec{b}$ lie in the plane spanned by e_1, e_2
- ▶ Write $\vec{c} = ue_1 + ve_2 + he_3$
 - ▶ Only the e_3 component carries $\vec{c}$ off the plane of the base, so call h the **height**
- ▶ The volume is the height times the area of the base:

$$\text{vol}(P(\vec{a}, \vec{b}, \vec{c})) = |h| |A(\vec{a}, \vec{b})|$$

- ▶ Again, we want to avoid the absolute values

Oriented Volume of a Parallelepiped



- Define the oriented volume of $P(\vec{a}, \vec{b}, \vec{c})$ to be

$$V(\vec{a}, \vec{b}, \vec{c}),$$

where

- $V(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$
- $|V(\vec{a}, \vec{b}, \vec{c})|$ is the volume of $P(\vec{a}, \vec{b}, \vec{c})$
- V is an antisymmetric multilinear function
- If $\vec{a}, \vec{b}$ lie in the plane spanned by $\mathbf{e}_1, \mathbf{e}_2$ and h is the $\mathbf{e}_3$ component of $\vec{c}$, this gives

$$V(\vec{a}, \vec{b}, \vec{c}) = h A(\vec{a}, \vec{b}),$$

with no absolute values

Oriented Volume is the Determinant of a 3-by-3 Matrix

- Suppose $v_1, v_2, v_3 \in V$, where, using Einstein notation,

$$\begin{aligned}[v_1 \quad v_2 \quad v_3] &= [e_k A_1^k \quad e_k A_2^k \quad e_k A_3^k] \\ &= [e_1 \quad e_2 \quad e_3] \begin{bmatrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^2 & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{bmatrix} \\ &= EA\end{aligned}$$

- The determinant of A is uniquely determined by the equation

$$V(v_1, v_2, v_3) = V(e_1, e_2, e_3) \det A$$

Multilinear Function on a Vector Space

- ▶ Let V be an n -dimensional vector space
 - ▶ It is OK to assume $V = \mathbb{R}^n$ and that a vector is a column matrix with n rows
- ▶ A function $A : V \times \cdots \times V \rightarrow \mathbb{R}$ of n vectors is **multilinear** if given vectors $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n, v, w \in V$ and scalars $a, b \in \mathbb{R}$,

$$\begin{aligned} A(t_1, \dots, t_{i-1}, av + bw, t_{i+1}, \dots, t_n) \\ = aA(t_1, \dots, t_{i-1}, v, t_{i+1}, \dots, t_n) \\ + bA(t_1, \dots, t_{i-1}, w, t_{i+1}, \dots, t_n) \end{aligned}$$

- ▶ In other words, if all but one input of A is held fixed, then A is a linear function of the remaining input

Permutations

- ▶ A permutation is a bijective map $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$
- ▶ Let S_n be the set of all permutations of order n
- ▶ A transposition is a permutation τ that switches two elements and leaves the others unchanged.

- ▶ Example: $\tau : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$, where

$$\tau(1) = 1, \tau(2) = 4, \tau(3) = 3, \tau(4) = 2$$

- ▶ Every permutation is a composition of transpositions
 - ▶ Example: The permutation $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ can be written as

$$\sigma = \tau_{13} \circ \tau_{12}, \text{ where}$$

$$\tau_{12}(1) = 2, \tau_{12}(2) = 1, \tau_{12}(3) = 3$$

$$\tau_{13}(1) = 3, \tau_{13}(2) = 2, \tau_{13}(3) = 1$$

Transpositions

- ▶ For $i \neq j$, the transposition $\tau_{ij} \in S_n$ is given by

$$\tau_{ij}(k) = \begin{cases} j & \text{if } k = i, \\ i & \text{if } k = j, \\ k & \text{otherwise.} \end{cases}$$

- ▶ Any permutation can be written as a composition of transpositions.
- ▶ The identity map, denoted ι , is the composition of zero transpositions.

Parity or Sign of a Permutation

- ▶ Given any permutation $\sigma \in S_n$, its parity or sign, which we will write as $\epsilon(\sigma)$, is defined to be
 - ▶ 1 if σ is the composition of an even number of transpositions
 - ▶ -1 if σ is the composition of an odd number of transpositions
- ▶ Easy consequences
 - ▶ If $\sigma \in S_n$ is a transposition, then $\epsilon(\sigma) = -1$
 - ▶ For any $\sigma_1, \sigma_2 \in S_n$, $\epsilon(\sigma_2 \circ \sigma_1) = \epsilon(\sigma_2)\epsilon(\sigma_1)$
 - ▶ If σ is the identity map, then $\epsilon(\sigma) = 1$
 - ▶ For any $\sigma \in S_n$, $\epsilon(\sigma^{-1}) = \epsilon(\sigma)$ because

$$\sigma = \tau_1 \circ \cdots \circ \tau_N \implies \sigma^{-1} = \tau_N \circ \cdots \circ \tau_1$$

Existence of Sign Function

- ▶ We have stated the properties that the sign function

$$\epsilon : S_n \rightarrow \{-1, 1\}$$

- ▶ Claim: There exists a unique function satisfying these properties
- ▶ This is the consequence of the following:
 - ▶ A permutation is never both the composition of an even number of transpositions and the composition of an odd number of transpositions
- ▶ There are straightforward elementary proofs of this
- ▶ There are also **many sophisticated proofs**

Maps of $\{1, \dots, n\}$ to Itself

- ▶ Let T_n denote the set of all maps

$$\phi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$$

- ▶ Observe that $S_n \subset T_n$
- ▶ We can extend the function $\epsilon : S_n \rightarrow \{-1, 1\}$ to a function

$$\epsilon : T_n \rightarrow \{-1, 0, 1\},$$

where, if $\phi \in S_n$, then $\epsilon(\phi)$ is as defined before and

$$\epsilon(\phi) = 0 \text{ if } \phi \notin S_n$$

Antisymmetric Function, Part 1

- ▶ A function $A : V \times \cdots \times V \rightarrow \mathbb{R}$ is **antisymmetric** if interchanging two arguments changes its sign.
- ▶ For a transposition τ_{ij} ,

$$A(v_{\tau_{ij}(1)}, \dots, v_{\tau_{ij}(n)}) = -A(v_1, \dots, v_n).$$

- ▶ If $v_i = v_j$ for some $i \neq j$, interchanging those arguments leaves the input unchanged, so

$$A(v_1, \dots, v_n) = -A(v_1, \dots, v_n) = 0.$$

Antisymmetric Function, Part 2

- ▶ If $\sigma = \tau_k \circ \cdots \circ \tau_1$, repeated antisymmetry gives

$$A(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^k A(v_1, \dots, v_n) = \epsilon(\sigma) A(v_1, \dots, v_n).$$

- ▶ If $\phi \in T_n \setminus S_n$, then two of the indices $\phi(1), \dots, \phi(n)$ agree, so

$$A(v_{\phi(1)}, \dots, v_{\phi(n)}) = 0.$$

- ▶ Thus, for every $\phi \in T_n$,

$$A(v_{\phi(1)}, \dots, v_{\phi(n)}) = \epsilon(\phi) A(v_1, \dots, v_n).$$

Antisymmetric Multilinear Function, Part 1

- ▶ Let $E = (e_1, \dots, e_n)$ be a basis of an n -dimensional vector space V .
- ▶ Given $v_1, \dots, v_n \in V$, write

$$v_k = e_1 b_k^1 + \dots + e_n b_k^n, \quad k = 1, \dots, n.$$

- ▶ If $A : V \times \dots \times V \rightarrow \mathbb{R}$ is multilinear, then

$$A(v_1, \dots, v_n) = \sum_{k_1=1}^n \dots \sum_{k_n=1}^n b_1^{k_1} \dots b_n^{k_n} A(e_{k_1}, \dots, e_{k_n}).$$

Antisymmetric Multilinear Function, Part 2

- ▶ If A is antisymmetric, every term with $k_i = k_j$ for some $i \neq j$ is zero.
- ▶ In every remaining term, $(k_1, \dots, k_n)$ is a permutation of $(1, \dots, n)$, so $k_j = \sigma(j)$ for some $\sigma \in S_n$.
- ▶ Therefore,

$$A(v_1, \dots, v_n) = \sum_{\sigma \in S_n} \epsilon(\sigma) b_1^{\sigma(1)} \cdots b_n^{\sigma(n)} A(e_1, \dots, e_n).$$

Antisymmetric Multilinear Function, Part 3

- ▶ For every $c \in \mathbb{R}$, there is a unique antisymmetric multilinear function A with n inputs satisfying

$$A(e_1, \dots, e_n) = c.$$

- ▶ In particular, there is a unique such function with $A(e_1, \dots, e_n) = 1$.
- ▶ If $A(e_1, \dots, e_n) = 1$, every antisymmetric multilinear function B with n inputs satisfies

$$B = cA, \quad c = B(e_1, \dots, e_n).$$

Definition of the Determinant

- ▶ Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$
- ▶ There is a unique antisymmetric multilinear function

$$D : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$$

with n inputs such that

$$D(e_1, \dots, e_n) = 1,$$

- ▶ The formula for D is

$$D(v_1, \dots, v_n) = \sum_{\phi \in T_n} \epsilon(\phi) b_{\phi(1)}^1 \cdots b_{\phi(n)}^n = \sum_{\sigma \in S_n} \epsilon(\sigma) b_{\sigma(1)}^1 \cdots b_{\sigma(n)}^n$$

where each $v_k = (b_k^1, \dots, b_k^n)$

- ▶ Let M be an n -by- n matrix with columns $C_1, \dots, C_n$
- ▶ The determinant of M is defined to be

$$\det M = D(C_1, \dots, C_n)$$